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Vibrations of an elastic cylindrical shell near the lowest cut-off frequency.

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A new asymptotic approximation of the dynamic equations in the 2D classical theory of thin elastic shells is established for a circular cylindrical shell. It governs long wave vibrations in the vicinity of the lowest cut-off frequency. At a fixed circumferential wave number the latter corresponds to the eigen frequency of in-plane vibrations of a thin almost inextensible ring. It is stressed that the well-known semi-membrane theory of cylindrical shells is not suitable for tackling a near-cut-off behaviour. The dispersion relation within the framework of the developed formulation coincides with the asymptotic expansion of the dispersion relation originating from full 2D shell equations. Asymptotic analysis also enables refining the geometric hypotheses underlying various adhoc setups, including the assumption on vanishing of shear and circumferential mid-surface deformations used in the semi-membrane theory. The obtained results may be of interest for dynamic modelling of elongated cylindrical thin walled structures, such as carbon nanotubes.

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1. Introduction

The theory of thin elastic shells, as an important branch of theoretical solid mechanics, widely exploits various approximate formulations. These are based on appropriate physical assumptions or originate from asymptotic expansions, e.g. see [1–5] and references therein. Each of the approaches has its own advantages. Sensible adhoc assumptions are often attractive due to the possibility of an immediate insight. At the same time their ranges of applicability are not always well defined. The asymptotic methodology is better mathematically justified but its implementation relies on a substantial preliminary analysis aimed to determine a correct scaling and often can't be completely formalized. Both of these approaches were extensively used within the framework of shell theory and it might be an impression that there is a little or no room for new developments in this area. In particular, it might be expected that all possible shortened forms of the classical 2D dynamic equations have been already derived [6]. However, recent achievements in nanotechnology [7–11] inspire revisiting of this problem. In contrast to macroscopic shells used in aerospace, mechanical and civil engineering, carbon nanotubes may have a large aspect ratio characterizing the relationship between the length and radius. The macroscopic shells with such aspect ratio are usually treated as 1D beams because of considerable overall (beam-type) flexibility. Design of modern nanodevices can require modelling of 1D beam-type vibrations with rigid cross-sections as well as 2D shell-type vibrations over a broad frequency band [12, 13].

Thorough analysis of the state of art in this area indicates that dynamic asymptotic theory for elongated cylindrical shells is not yet established. In the limiting setup of an infinite shell the most pronounced long wave vibrations at a fixed circumferential wave number occur near the lowest cut-off frequency coinciding with the eigen-frequency of a thin almost inextensible ring. In this case the dispersion relation associated with a consistent approximate long wave theory has to coincide with the appropriate near-cut-off expansion of the exact dispersion relation in the full classical shell theory. The well-known semi-membrane shell theory, e.g. see [1, 2, 14], originally attributed to statics, violates this condition. The aim of the paper is to derive an asymptotic theory governing dynamic behaviour of a circular cylindrical shell in the vicinity of the lowest cut-off frequency.

It is interesting that near-cut-off asymptotic expansions were previously studied only for various setups of high frequency thickness shell vibrations, e.g. see [4, 5, 15–17]. The developed long wave approximations have been applied to trapped mode analysis [18–20].

The paper is organised as follows. The problem is formulated in Sect. 2 starting from the Sanders-Koiter version of the classical shell theory, e.g. see [21]. In Sect. 3 the associated dispersion relation is derived along with its asymptotic expansion valid in the vicinity of the lowest cut-off frequency. The latter has two limiting forms depending on the ratio of the longitudinal wavelength and the relative thickness of the shell. In the next section a straightforward asymptotic procedure is adapted for the Sanders-Koiter equations of motion. The established fourth-order ordinary differential equation for the transverse displacement has two shortened forms. The first of them is nothing than the semi-membrane theory, whereas the second one is given by a second-order equation typical of near-cut-off approximations, e.g. see [4]. The dispersion relations for these two shortened equations coincide with the expansions of the exact dispersion relations in Sect. 3. It is shown in Sect. 5 that the asymptotic considerations of the previous section suggest rather sophisticated geometric hypotheses underlying the obtained equation. They refine the assumption of zero shear and circumferential mid-surface deformations underlying the semi-membrane theory. It is also demonstrated in this section that the earlier proposed adhoc equation [22], based on the hypotheses in the semi-membrane theory but also taking into account some extra terms in the related energy functional, is virtually identical to its asymptotic counterpart to within the values of constant coefficients.

2. Equations of motion

Consider linear vibrations of a circular cylindrical shell (see Fig.1) starting from the Sanders-Koiter version of the 2D classical theory, e.g. [21]. This version allows elegant variational formulation along with natural generalization for nonlinear problems, see also [23] for more detail. Let us introduce the dimensionless coordinates ξ and φ, where $\xi = x/L$ is the longitudinal co-ordinate related to typical wavelength (or length) L and φ is the azimuthal angle. Denote by u, v and w longitudinal, circumferential and radial mid-surface displacements related to the mid-surface radius R. Also define dimensionless time using the scaling $t_0 = 1/\sqrt{E/\rho R^2(1-\nu^2)}$, where E is the Young modulus, ρ is mass density, ν is the Poisson ratio and h is the thickness of the shell.

The inverse aspect ratio $\alpha = R/L$ and the relative thickness $\beta = h/R$ are two major geometric parameters governing shell vibrations. The parameter β is small in all 2D thin shell theories. The parameter α is also small for elongated shells. Below we always assume that

$$\alpha \ll 1, \qquad \beta \ll 1. \tag{2.1}$$

The Hamilton function within the framework of the Sanders-Koiter shell theory can be written as

$$\begin{aligned} H = &\frac{1}{2}\int_0^1\int_0^{2\pi}\left(\dot{u}^2+\dot{v}^2+\dot{w}^2\right)d\varphi d\xi + \\ &+\frac{1}{2}\int_0^1\int_0^{2\pi}\left(\varepsilon_\xi^2+\varepsilon_\varphi^2+2\nu\varepsilon_\xi\varepsilon_\varphi+\frac{\varepsilon_{\xi\varphi}^2(1-\nu)}{2}\right)d\varphi d\xi + \\ &+\frac{\beta^2}{24}\int_0^1\int_0^{2\pi}\left(\kappa_\xi^2+\kappa_\varphi^2+2\nu\kappa_\xi\kappa_\varphi+\frac{\kappa_{\xi\varphi}^2(1-\nu)}{2}\right)d\varphi d\xi, \end{aligned} \tag{2.2}$$

where ε_ξ, ε_φ and $\varepsilon_{\xi\varphi}$ are longitudinal, circumferential and shear deformations of the mid-surface; κ_ξ and κ_φ are curvatures of the mid-surface in the longitudinal and circumferential directions; $\kappa_{\xi\varphi}$ is torsion of the mid-surface. Here dot denotes a partial derivative with respect to the dimensionless time $\tau = t/t_0$.

Kinematic relations expressing the deformations, curvatures and torsion in terms of the displacements are

$$\varepsilon_\xi = \alpha\frac{\partial u}{\partial \xi}, \qquad \varepsilon_\varphi = \frac{\partial v}{\partial \varphi} + w, \qquad \varepsilon_{\xi\varphi} = \frac{\partial u}{\partial \varphi} + \alpha\frac{\partial v}{\partial \xi}, \tag{2.3}$$

and

$$\kappa_\xi = -\alpha^2\frac{\partial^2 w}{\partial \xi^2}, \quad \kappa_\varphi = \frac{\partial v}{\partial \varphi} - \frac{\partial^2 w}{\partial \varphi^2},$$
$$\kappa_{\xi\varphi} = -2\alpha\frac{\partial^2 w}{\partial \xi \partial \varphi} + \frac{3\alpha}{2}\frac{\partial v}{\partial \xi} - \frac{1}{2}\frac{\partial u}{\partial \varphi}.$$

Variation of the functional H with respect to the displacements u, v and w after substitution of (2.3) into (2.2) leads to three partial differential equations

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial \tau^2} - \alpha^2 \frac{\partial^2 u}{\partial \xi^2} - \frac{(48+\beta^2)(1-\nu)}{96}\frac{\partial^2 u}{\partial \varphi^2} - \alpha\nu \frac{\partial w}{\partial \xi} - \\
&\quad - \frac{\alpha\big(16(1+\nu)-\beta^2(1-\nu)\big)}{32}\frac{\partial^2 v}{\partial \xi \partial \varphi} - \frac{\alpha\beta^2(1-\nu)}{24}\frac{\partial^3 w}{\partial \xi \partial \varphi^2} = 0, \\[2ex]
&\frac{\partial^2 v}{\partial \tau^2} - \frac{\alpha^2(16+3\beta^2)(1-\nu)}{32}\frac{\partial^2 v}{\partial \xi^2} - \frac{12+\beta^2}{12}\frac{\partial^2 v}{\partial \varphi^2} - \frac{\partial w}{\partial \varphi} - \\
&\quad - \frac{\alpha\big(16(1+\nu)-\beta^2(1-\nu)\big)}{32}\frac{\partial^2 u}{\partial \xi \partial \varphi} + \frac{\alpha^2\beta^2(3-\nu)}{24}\frac{\partial^3 w}{\partial \xi^2 \partial \varphi} + \frac{\beta^2}{12}\frac{\partial^3 w}{\partial \varphi^3} = 0, \\[2ex]
&\frac{\partial^2 w}{\partial \tau^2} + w + \frac{\alpha^2\beta^2}{6}\frac{\partial^4 w}{\partial \xi^2 \partial \varphi^2} + \frac{\alpha^4\beta^2}{12}\frac{\partial^4 w}{\partial \xi^4} + \frac{\beta^2}{12}\frac{\partial^4 w}{\partial \varphi^4} + \alpha\nu\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \varphi} + \\
&\quad + \frac{\alpha\beta^2(1-\nu)}{24}\frac{\partial^3 u}{\partial \xi \partial \varphi^2} - \frac{\alpha^2\beta^2(3-\nu)}{24}\frac{\partial^3 v}{\partial \xi^2 \partial \varphi} - \frac{\beta^2}{12}\frac{\partial^3 v}{\partial \varphi^3} = 0.
\end{aligned}
\tag{2.4}
$$

On separating the azimuthal and time variables in (2.4) by the formulae

$$
u = U\cos n\varphi \cos \omega\tau, \quad v = V \sin n\varphi \cos \omega\tau, \quad w = W \cos n\varphi \cos \omega\tau, \tag{2.5}
$$

where n is circumferential wave number, ω is vibration frequency, and $U(\xi)$, $V(\xi)$ and $W(\xi)$ are vibration amplitudes, we arrive at ordinary differential equations

$$
\begin{aligned}
&-\omega^2 U + \frac{n^2(48+\beta^2)(1-\nu)}{96}U - \alpha^2\frac{d^2 U}{d\xi^2} - \frac{\alpha n\big(16(1+\nu)-\beta^2(1-\nu)\big)}{32}\frac{dV}{d\xi} - \\
&\quad - \frac{\alpha\big(24\nu - n^2\beta^2(1-\nu)\big)}{24}\frac{dW}{d\xi} = 0, \\[2ex]
&-\omega^2 V + \frac{n^2(12+\beta^2)}{12}V + \frac{n(12+\beta^2 n^2)}{12}W - \frac{\alpha^2(16+3\beta^2)(1-\nu)}{32}\frac{d^2 V}{d\xi^2} + \\
&\quad + \frac{\alpha n\big(16(1+\nu)-\beta^2(1-\nu)\big)}{32}\frac{dU}{d\xi} - \frac{\alpha^2\beta^2 n(3-\nu)}{24}\frac{d^2 W}{d\xi^2} = 0, \\[2ex]
&-\omega^2 W + \frac{12+n^4\beta^2}{12}W + \frac{n(12+n^2\beta^2)}{12}V - \frac{\alpha^2\beta^2 n^2}{6}\frac{d^2 W}{d\xi^2} + \frac{\alpha^4\beta^2}{12}\frac{d^4 W}{d\xi^4} + \\
&\quad + \frac{\alpha\big(24\nu - n^2\beta^2(1-\nu)\big)}{24}\frac{dU}{d\xi} - \frac{\alpha^2\beta^2 n(3-\nu)}{24}\frac{d^2 V}{d\xi^2} = 0.
\end{aligned}
\tag{2.6}
$$

3. Dispersion relation

For an infinite shell or in case of periodic boundary conditions $V(0) = V(2\pi) = W(0) = W(2\pi) = U_\xi(0) = U_\xi(2\pi) = 0$ we may set (k is normalized longitudinal wave number)

$$
U(\xi) = U_0 \cos k\xi, \quad V(\xi) = V_0 \sin k\xi, \quad W(\xi) = W_0 \sin k\xi, \tag{3.1}
$$

having three linear algebraic equations which are compatible provided that

$$\begin{aligned}
&\omega^6 - \tfrac{\omega^4}{96}\Big(\alpha^2k^2\big(\beta^2(8\alpha^2k^2 - 9\nu + 9) - 48(\nu-3)\big) + \\
&+ n^2\big(\beta^2(16\alpha^2k^2 - \nu + 9) - 48(\nu-3)\big) + 8\beta^2n^4 + 96\Big) - \\
&- \tfrac{\omega^2}{1152}\Bigg(n^4\Big(\beta^2\Big(\beta^2(\nu-1) + \alpha^2k^2\big(\beta^2(\nu-1)(2\nu+1) + 144(\nu-3)\big) + 60\nu + 132\Big) + \\
&+ 576(\nu-1)\Big) + 3\alpha^2k^2\Big(12(\nu-1)(3\beta^2+32\nu+48) + \alpha^4\beta^2k^4\big(3\beta^2(\nu-1) + 16(\nu-3)\big) + \\
&+ 12\alpha^2k^2(\nu-1)(3\beta^2+16)\Big) + n^2\Big(12\big(\beta^2(\nu-9) + 48(\nu-1)\big) + \alpha^4\beta^2k^4\big(\beta^2(\nu-1)(2\nu+9) + \\
&+ 144(\nu-3)\big) + 24\alpha^2\beta^2k^2(3\nu^2+45\nu-44)\Big) + \beta^2n^6\big(\beta^2(\nu-1) + 48(\nu-3)\big)\Bigg) + \\
&+ \tfrac{1-\nu}{13824}\Big(- 36\alpha^8\beta^2k^8(3\beta^2+16) - 192\alpha^6\beta^2k^6n^2(\beta^2+12) + \\
&+ \alpha^4k^4\big(6912(\nu^2-1) + \beta^6n^4(\nu^2-1) + 72\beta^4n^2(\nu^2-n^2) - 432\beta^2(-3\nu^2+8n^4-8n^2+3)\big) - \\
&- 2304\alpha^2k^2\beta^2n^2(n^2-1)^2 - 12\beta^2n^4(\beta^2+48)(n^2-1)^2\Big) = 0.
\end{aligned} \tag{3.2}$$

This shows that the frequency spectrum of the shell consists of three branches, e.g. see [4, 5] for greater detail. The lowest of them is associated with bending of the mid-surface strongly effected by shell curvature, see also [6]. These low-frequency bending vibrations are the main focus of the paper. Two other branches correspond to extension and shear motions along the mid-surface and are less affected by curvature. As an illustration three dispersion curves are shown in Fig. 2 for $n = 2$ and $\nu = 0.2$; in this figure $k_\alpha = \alpha k$. The curve of interest is plotted with a bold line.

The equations for the cut-off frequency of the aforementioned lowest branch follow from (2.6) at $U = \dfrac{dV}{d\xi} = \dfrac{dW}{d\xi} = 0$. They are

$$\begin{aligned}
&\left(\tfrac{n^2(12+\beta^2)}{12} - \omega^2\right)V + \tfrac{n(12+\beta^2n^2)}{12}W = 0, \\
&\tfrac{n(12+\beta^2n^2)}{12}V + \left(\tfrac{12+\beta^2n^4}{12} - \omega^2\right)W = 0.
\end{aligned} \tag{3.3}$$

The latter govern free in-plane vibrations of an almost inextensible thin ring (see Fig. 3) with the eigen frequency $\omega = \omega_*$ given by

$$\omega_*^2 = \frac{(1+n^2)(12+\beta^2n^2)}{24}\left(1 + \sqrt{1 - 48\left(\frac{\beta n(1-n^2)}{(1+n^2)(12+\beta^2n^2)}\right)^2}\right). \tag{3.4}$$

Obviously, this frequency satisfies the dispersion relation (3.2) at $k = 0$.

Exact analysis of (3.2) is a rather sophisticated problem. At the same time for the lowest branch of interest the asymptotic behaviour of the related root of (3.2) under assumptions (2.1) is expressed by the relatively simple formula ($k \sim 1$ and $n \sim 1$)

$$\omega^2 = \beta^2\Omega_0^2 + \beta^4\Omega_1^2 + \alpha^2\beta^2\delta k^2 + \alpha^4\gamma k^4 + \ldots, \tag{3.5}$$

where

$$\Omega_0^2 = \frac{n^2(1-n^2)^2}{12(1+n^2)}, \qquad \Omega_1^2 = -\frac{n^6(1-n^2)^2}{36(1+n^2)^3}, \tag{3.6}$$

and

$$\delta = \frac{(1-n^2)^2(2n^2+1-2\nu)}{12(1+n^2)^2}, \qquad \gamma = \frac{1-\nu^2}{n^2(1+n^2)}. \tag{3.7}$$

The formula (3.6) may be also presented as a near-cut-off expansion, i.e.

$$\omega^2 - \omega_*^2 = \alpha^2\beta^2\delta k^2 + \alpha^4\gamma k^4 + \dots \tag{3.8}$$

Here the cut-off frequency (3.4) can be approximated by a two-term asymptotic formula as $\beta \ll 1$. It is

$$\omega_*^2 = \beta^2\Omega_0^2 + \beta^4\Omega_1^2 + \dots. \tag{3.9}$$

Two terms are retained in the right hand side of (3.8) due to a two-parametric nature of the problem noted above. At $\alpha \ll \beta$ we get the leading order near-cut-off expansion

$$\omega^2 - \omega_*^2 = \alpha^2\beta^2\delta k^2 + \dots. \tag{3.10}$$

In case $\alpha \gg \beta$ (3.9) may be simplified to

$$\omega^2 - \omega_*^2 = \alpha^4\gamma k^4 + \dots, \tag{3.11}$$

with $\omega_*^2 = \beta^2\Omega_0^2 + \dots$.

It is worth noting that at $\alpha \sim \beta^{\frac{1}{2}}$ all the terms in (3.11) are of the same asymptotic order as it is usually assumed in the dynamic semi-membrane theory [24].

Numerical results are presented in Fig. 4 for $n=2$ and $n=3$. The logarithmic deviation from the cut-off frequency $\Delta = \lg \beta^{-4}(\omega^2 - \omega_*^2)$ is plotted versus the ratio $\alpha/\beta = R^2/Lh$ at $k=1$ and $\nu = 0.2$. The values Δ for the full dispersion relation at $\beta = 0.01$ (3.2) and its asymptotic expansions (3.8), given by solid line, are not distinguishable in this figures. The curves corresponding to formulae (3.10) and (3.11) are plotted with dotted and dash-dotted lines, respectively. As might be expected, the semi-membrane theory (see formula (3.11)) is valid outside a narrow vicinity of the cut-off frequency, in which formula (3.10) demonstrates a better accuracy. At the same time the two-term formula (3.8) picks up uniform near-cut-off behaviour.

4. Asymptotic derivation

For the sake of simplicity, we set in (2.6) $\alpha = \beta \ll 1$. In this case both terms in the right hand side of the asymptotic expansion (12) are of the same order. Let us specify asymptotic series

$$\begin{aligned} V &= v_0 + \beta^2 v_1 + \beta^4 v_2 + \dots, \\ W &= w_0 + \beta^2 w_1 + \beta^4 w_2 + \dots, \\ U &= \beta(u_0 + \beta^2 u_1 + \beta^4 u_2 + \dots) \end{aligned} \tag{4.1}$$

and

$$\omega^2 = \beta^2(\omega_0 + \beta^2\omega_1 + \beta^4\omega_2 + \dots).$$

At leading order, we get from second and third equations in (2.6)

$$n v_0 + w_0 = 0. \tag{4.2}$$

At the same time first equation in (2.6) becomes

$$(\nu - 1)n^2 u_0 + (\nu + 1)n v_0' + 2\nu w_0' = 0. \tag{4.3}$$

Here and below prime denotes differentiation in ξ. From (4.2) and (4.3) we immediately obtain

$$v_0 = -\frac{w_0}{n}, \qquad u_0 = -\frac{w_0'}{n^2}, \tag{4.4}$$

meaning that circumferential and shear deformations can be neglected at leading order. Such assumption is typical of the semi-membrane shell theory, e.g. see [1, 2, 14].

At first order, we have from second and third equations in (2.6)

$$\begin{aligned} nv_1 + w_1 &= \frac{\nu w_0^{''}}{n^2} + \left(\frac{1-n^2}{12} - \frac{\omega_0^2}{n^2}\right) w_0, \\ \frac{\nu}{n^2} w^{''} + \left(\omega_0^2 + \frac{n^2(1-n^2)}{12}\right) w_0 - nv_1 - w_1 &= 0. \end{aligned} \tag{4.5}$$

We get from the compatibility of these equations $\omega_0 = \Omega_0$, where Ω_0 is given by (3.6). As a result, each of them takes the form

$$v_1 = -\frac{w_1}{n} + L(w_0), \tag{4.6}$$

where the operator

$$L(w_0) = \frac{\nu w_0^{''}}{n^3} + \frac{n(1-n^2)}{6(1+n^2)} w_0 \tag{4.7}$$

corresponds to the deviation of the sought for displacement field from the behaviour predicted by the semi-membrane theory.

Now, we consider the first equation in (2.6), which at first order is given by

$$(\nu-1)n^2 u_1 + (\nu+1)nv_1^{'} + 2\nu w_1^{'} = \frac{2}{n^2} w_0^{'''} + \frac{(1-n^2)(-3n^2+1+\nu(n^2+1))}{12(n^2+1)} w_0^{'}. \tag{4.8}$$

By making use of (4.6) we transform the last equation to

$$u_1 = -\frac{w_1^{'}}{n^2} + M(w_0), \tag{4.9}$$

where the opeator $M(w_0)$ is expressed as

$$M(w_0) = \frac{1}{(1-\nu)n^2}\left(\frac{\nu(\nu+1)-2}{n^2} w_0^{'''} - \frac{(1-n^2)(-5n^2+1+\nu(1-n^2))}{12(n^2+1)} w_0^{'}\right). \tag{4.10}$$

At second order, we obtain from second and third equations in (2.6), respectively

$$\begin{aligned} nv_2 + w_2 = &\left(\frac{(1-\nu)(n^2-1)}{8n^2} + \frac{\nu}{12}\right) w_0^{''} - \frac{\omega_1^2}{n^2} w_0 + \frac{1-\nu}{2n} L^{''}(w_0) + \\ &+ \frac{n^3(n^2-3)}{12(1+n^2)} L(w_0) - \frac{1+\nu}{2} M^{'}(w_0) + \frac{\nu}{n^2} w_1^{''} - \frac{n^2(n^2-1)}{6(1+n^2)} w_1, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &\left(\frac{1-n^2}{6} - \frac{\nu}{12}\right) w_0^{''} - \omega_1^2 w_0 + \frac{n^3}{12} L(w_0) + \nu M^{'}(w_0) - \\ &- \frac{\nu}{n^2} w_1^{''} + \frac{n^2(n^2-1)}{6(1+n^2)} w_1 + nv_2 + w_2 = 0. \end{aligned} \tag{4.12}$$

Then, on substituting (4.11) into (4.12), we arrive at

$$\begin{aligned} &\frac{n^2-1}{2}\left(\frac{1-\nu}{4n^2} - \frac{1}{3}\right) w_0^{''} - \frac{n^2+1}{n^2} \omega_1^2 w_0 + \\ &+ \frac{1-\nu}{2n} L^{''}(w_0) + \frac{n^3(n^2-1)}{6(n^2+1)} L(w_0) - \frac{1-\nu}{2} M^{'}(w_0) = 0, \end{aligned} \tag{4.13}$$

and finally,

$$\gamma w_0^{''''} - \delta w_0^{''} + \beta^{-4}(\omega_*^2 - \omega^2) w_0 = 0, \tag{4.14}$$

where the constant coefficients δ, γ and ω_* are given by (3.6) and (3.9). In the original variables x and t the last equation becomes

$$\gamma R^2 \frac{\partial^2 W}{\partial x^4} - \frac{\delta h^2}{R^2}\frac{\partial^2 W}{\partial x^2} + \frac{\omega_*^2}{R^2} W + \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0, \tag{4.15}$$

where

$$c = \sqrt{\frac{E}{\rho(1-\nu^2)}}.$$

Its range of validity is $\alpha \sim \beta \ll 1$ $(R \sim \sqrt{Lh})$. At $\alpha \ll \beta$ $(R \ll \sqrt{Lh})$ the derived equation may be reduced to a typical near-cut-off second order equation, e.g. see [4]. It is

$$\frac{\delta h^2}{R^2}\frac{\partial^2 W}{\partial x^2} - \frac{\omega_*^2}{R^2}W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0, \tag{4.16}$$

corresponding to the shortened dispersion relation (3.10).

The semi-membrane shell dynamic equation [24]

$$\gamma R^2 \frac{\partial^2 W}{\partial x^4} + \frac{\omega_*^2}{R^2}W + \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0 \tag{4.17}$$

is valid at $\alpha \gg \beta$ $(R \gg \sqrt{Lh})$. As we have already mentioned in the Section 2 (see also the dispersion relation (3.11)), all the terms in the last equation are of the same order at $\alpha \sim \beta^{\frac{1}{2}}$ $(R \sim L^{\frac{2}{3}} h^{\frac{1}{3}})$.

5. Geometric hypotheses

The derivation in the previous section, see (4.6), (4.7) and (4.9), (4.10), suggests that the equation (4.15) should also follow from the geometric hypotheses

$$V + \frac{1}{n}W = \alpha^2 \frac{\nu}{n^3}\frac{d^2 W}{d\xi^2} + \beta^2 \frac{n(1-n^2)}{6(1+n^2)}W \tag{5.1}$$

and

$$U + \frac{\alpha}{n^2}\frac{dW}{d\xi} = \frac{1}{(1-\nu)n^2}\left(\alpha^3 \frac{\nu(\nu+1)-2}{n^2}\frac{d^3 W}{d\xi^3} - \alpha\beta^2 \frac{(1-n^2)(-5n^2+1+\nu(1-n^2))}{12(n^2+1)}\frac{dW}{d\xi}\right).$$

These refine the above mentioned famous formula

$$U + \frac{\alpha}{n^2}\frac{dW}{d\xi} = 0, \qquad V + \frac{1}{n}W = 0 \tag{5.2}$$

for vanishing circumferential and shear mid-surface deformations in the semi-membrane theory of shells, e.g. see [1, 24].

On introducing (5.1) into the Hamiltonian (2.2) taking into account formula (2.3) and integrating over the angle φ, we get by varying it in W an equation identical to (4.15) to within asymptotically secondary terms in α and β. It is interesting that a similar manipulation with the simplified relations (5.2) results at leading order in the equation

$$\frac{R^2}{n^2(n^2+1)}\frac{\partial^4 W}{\partial x^4} - \frac{h^2(n^2-1)(2n^4-n^2-1+2\nu(n^2+1))}{12R^2(1+n^2)^2}\frac{\partial^2 W}{\partial x^2} + \\ + \frac{n^2(n^2-1)^2}{12R^2(n^2+1)}W + \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0, \tag{5.3}$$

which is also not far away from the asymptotically justified equation (4.15), cf the coefficients in (5.3) and those in (4.15) given by (3.6). Indeed, almost all of them, except Ω_1, coincide at $\nu = 0$. The last equation is asymptotically identical to that in [22].

6. Concluding remarks

An asymptotic theory valid in the vicinity of the lowest cut-off frequency is derived for a circular cylindrical shell. For a fixed circumferential wave number its shortened forms are given by the conventional fourth order equation in the semi-membrane theory and a second-order equation typical for near-cut-off approximations.

Consistency of the proposed theory is illustrated by comparison with the dispersion relation in the Sanders-Koiter theory. The asymptotic behaviour of shell displacements established in the paper enables refining the traditional assumption on zero shear and circumferential mid-surface deformations underlying both the semi-membrane theory and the more general adhoc theory in [22].

The developed approach allows various extensions and generalisations. In particular, asymptotic derivation of the long wave theory in question from the 3D dynamic equations in linear elasticity seems to be of obvious interest. The effect of geometric non-linearity also can be incorporated in order to justify the analogues adhoc non-linear equation in [22]. Finally, we mention applications dealing with localised vibrations, including those caused by shape imperfections as well as shell edge waves, see review article [25] and references therein.

The proposed theory appears to be of interest for predicting the vibration spectra of nanotubes over a broad range of geometric parameters for various boundary conditions specified at the edges. The small parameter α expressing the inverse aspect ratio of a nanotube may approach the values of order 0.01, here and below see [26]. Another small parameter β is evaluated as the ratio of extensional and bending nanotube stiffness rather than relative thickness of a macro shell. Its values are usually within the interval $0.05 \div 0.1$. Typical for nanotubes boundary conditions include, in particular, a free edge as well as an edge cupped with a fullerene hemisphere. In this case the forth-order differential equation (4.15) is subject to two conditions at each edge. For example, conditions on longitudinal and shear stress resultants are imposed at a free edge [1, 2].

Ethics statement. This work does not involve any aspects such as collection of human data or such.

Data accessibility. This work does not have any experimental data.

Conflict of interests. We have no competing interests.

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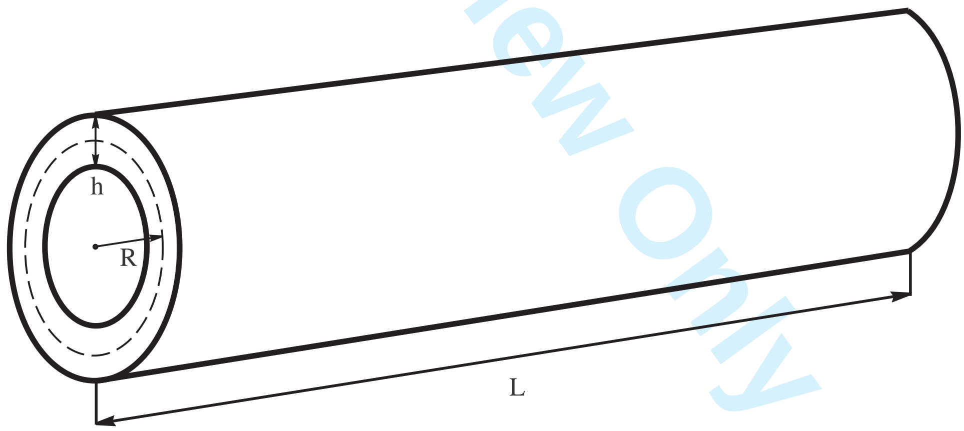

Figure 1. Circular cylindrical shell.

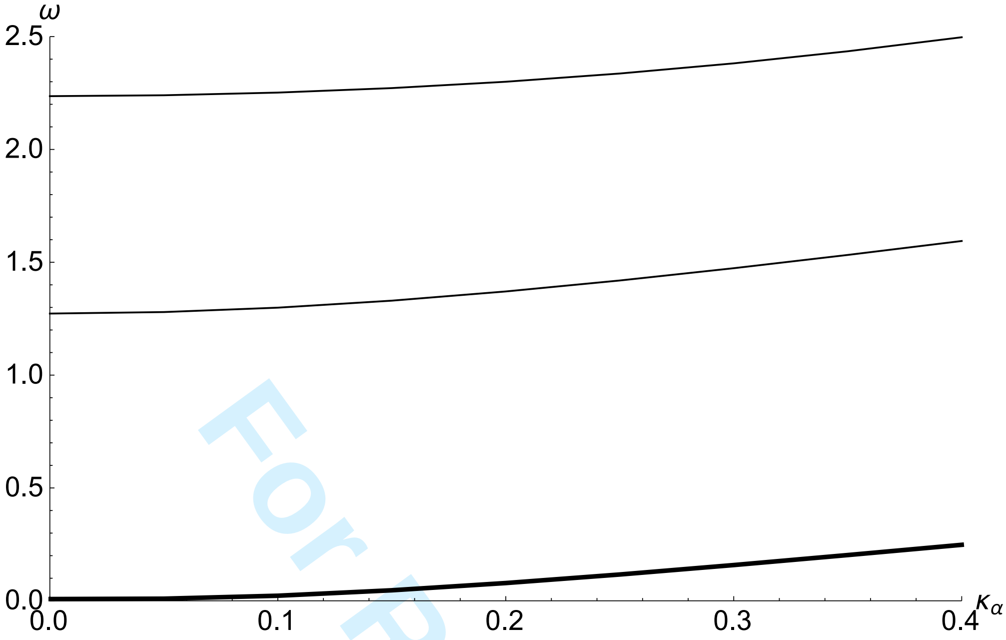

Figure 2. Dispersion curves for equation (3.2), $n = 2$. The bending branch is plotted with bold line.

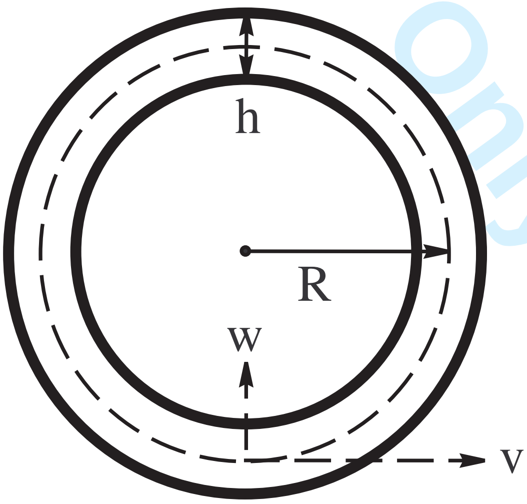

Figure 3. Thin almost inextensible ring.

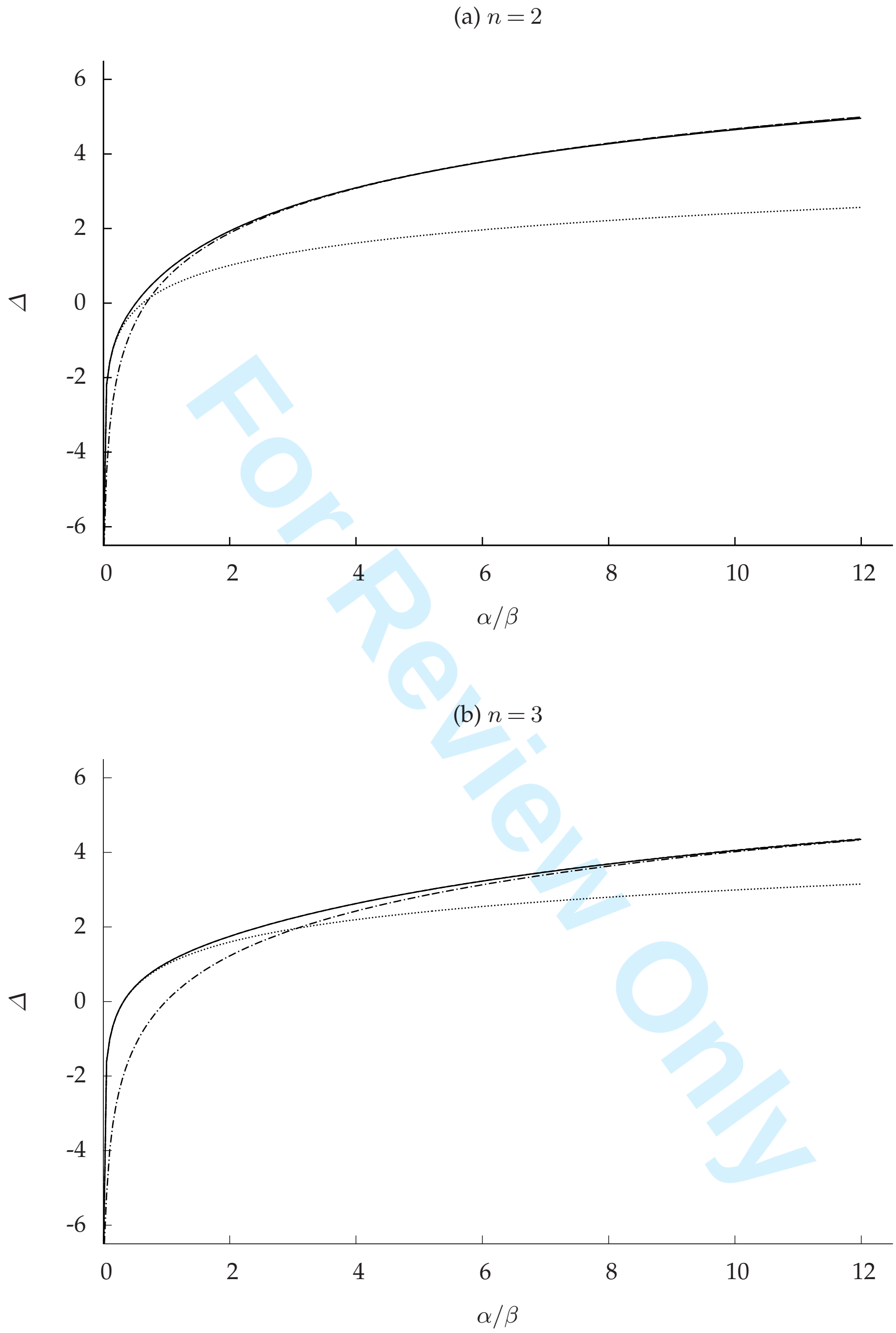

Figure 4. Dispersion curves for the Sanders-Koiter theory and two-term expansion (3.8) (solid line), traditional one-term expansion (3.10) (dotted line) and the semi-membrane theory (3.11) (dashed-dotted line).