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Dispersion of elastic waves in laminated glass

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Abstract

Elastic sandwich-type structures with high-contrast material and geometrical properties have numerous applications in modern engineering, including, in particular, laminated glass, photovoltaic panels, precipitator plates in gas filters, etc. Multi-parametric modelling of such structures assumes taking into consideration various types of contrast in stiffness, density and thickness. The present contribution is concerned with analysis of low-frequency dispersion of elastic waves in case of an antisymmetric motion of a three-layered symmetric plate, modelling laminated glass. The conditions on material and geometrical parameters, leading to the lowest non-zero thickness shear resonance frequency tending to zero, are formulated. In this case the dispersion relation possesses two low-frequency modes instead of a single fundamental low-frequency mode, which is typical for a homogeneous plate. A two-mode uniform asymptotic approximation is constructed, along with local approximations for the fundamental mode and the first shear harmonic.

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1. Introduction

Laminated glass is a sandwich-type structure consisting of two stiff glass layers separated by a soft PVB polymeric interlayer. It is widely used in automotive industry for manufacturing of windscreens as well as in civil engineering for architectural glazing. An important feature of laminated glass is a high contrast in mechanical properties of the layers. The elastic modulus of the interlayer is in several orders of magnitude lower than that of the skin glass layers, see e.g. [1]. In this case various engineering plate theories cannot be readily exploited, see discussions in [1], [2], [3]. Geometry of laminated glass also exhibits a considerable contrast, since the interlayer is much thinner than the outer layers. The presence of contrast, as shown in [4], [5], and [6] for inhomogeneous strings, rods, and sandwich plates, respectively, may result in a small first shear resonance frequency. Thus, the fundamental mode and first harmonic may both appear over the low-frequency band. This motivates derivation of two-mode approximations of the exact dispersion relation, see e.g. [7].

Below we extend the approach in [6] to laminated glass. First, we determine the range of material and geometric parameters enabling the lowest shear thickness resonance to be small. Then, we construct uniform two-mode low-

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frequency approximation for antisymmetric dispersion relation, valid both for the fundamental mode and for the first harmonic. The local one-mode approximations are also presented for these two modes. Numerical illustrations demonstrate the efficiency of the proposed approach.

2. Governing equations

Consider a symmetric three-layered elastic plate of infinite lateral extent with core and skin layers of thickness $2h_c$ and h_s, respectively, see Figure 1. For the sake of simplicity, we employ the plane strain assumption.

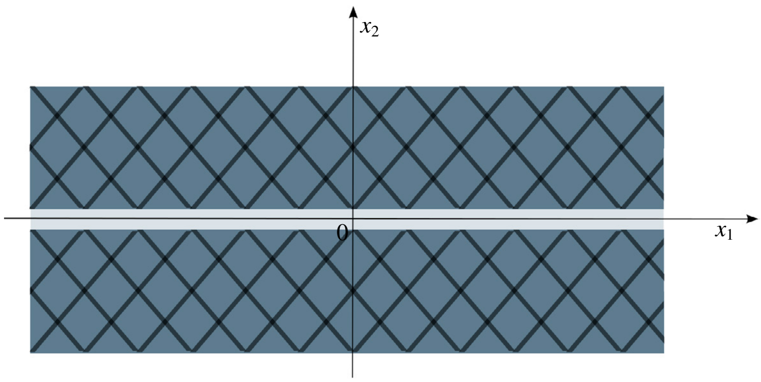

Fig. 1: A symmetric three-layered plate

The equations of motion for the core and skin layers are given by

$$\sigma^q_{ij,j} = \rho_q \ddot{u}^q_i, \quad i = 1, 2, \quad q = c, s \tag{1}$$

where $q = c$ and $q = s$ for the core and skin layers, respectively; summation over the repeated indices is assumed. Hereinafter σ^q_{ij} are stresses, u_i are displacements, ρ_q are volume mass densities.

The constitutive relations for a linearly isotropic elastic material are given by

$$\sigma^q_{ij} = \lambda_q \varepsilon^q_{kk} \delta_{ij} + 2\mu_q \varepsilon^q_{ij}, \tag{2}$$

with

$$\varepsilon^q_{ij} = \frac{1}{2}(u^q_{i,j} + u^q_{j,i}), \quad q = c, s, \tag{3}$$

where ε^q_{ij} are strains, and λ_q and μ_q are the Lamé parameters.

The traction free boundary conditions

$$\sigma^s_{12} = \sigma^s_{22} = 0 \tag{4}$$

are imposed along the faces $x_2 = \pm(h_c + h_s)$, together with the continuity conditions

$$\sigma^c_{12} = \sigma^s_{12}, \quad \sigma^c_{22} = \sigma^s_{22}, \quad u^c_1 = u^s_1, \quad u^c_2 = u^s_2 \tag{5}$$

along the interfaces $x_2 = \pm h_c$.

Let us define the dimensionless frequency Ω and wave number K as

$$\Omega = \frac{\omega h_c}{c_{2c}}, \qquad K = kh_c, \tag{6}$$

and introduce the dimensionless parameters

$$h = \frac{h_s}{h_c}, \qquad \varepsilon = \frac{\mu_c}{\mu_s}, \qquad r = \frac{\rho_c}{\rho_s}, \tag{7}$$

expressing the contrast in thickness, stiffness, and density of the core and skin layers.

In this case the dispersion relation for antisymmetric modes of the plate may be derived through a standard procedure, see e.g. [7]. The result is presented in Appendix A.

3. Asymptotic approach

Similarly to [5] we derive the conditions on material and geometrical parameters providing the lowest shear cut-off frequency being small, see also [6]

$$r \ll h \ll \varepsilon^{-1}, \tag{8}$$

with the lowest eigenvalue given at leading order by

$$\Omega_{sh} \approx \left(\frac{r}{h}\right)^{1/2} \ll 1. \tag{9}$$

For the global long-wave low-frequency regime, characterised by polynomial variation of the sought for solution across the plate thickness we require

$$K(1+h) \ll 1, \qquad \Omega\left(1 + h\left(\frac{\varepsilon}{r}\right)^{1/2}\right) \ll 1. \tag{10}$$

Under these assumptions we derive the long-wave low-frequency asymptotic expansions of the transcendental Rayleigh-Lamb equation (A.1). Expanding the hyperbolic functions in Rayleigh-Lamb equation (A.1) we arrive at

$$\gamma_1\Omega^2 + \gamma_2 K^4 + \gamma_3 K^2\Omega^2 + \gamma_4 K^6 + \gamma_5\Omega^4 + \gamma_6 K^4\Omega^2 + \gamma_7 K^8 + \gamma_8 K^2\Omega^4 + \gamma_9 K^6\Omega^2 + \gamma_{10} K^{10} + ... = 0, \tag{11}$$

where the coefficients γ_i are given in [6].

Typical parameters for laminated glass suggest the following relations, see e.g. [1],

$$h \sim \varepsilon^{-1/4}, \quad r \sim 1, \tag{12}$$

with the small parameter $\varepsilon \ll 1$. In this case the leading order behaviour of the coefficients γ_i in (11) is given by $\gamma_i = G_i\varepsilon^{\delta}$, where $G_i \sim 1$ and δ is a constant. In addition, the conditions (10) become $\Omega \ll 1$ and $K \ll \varepsilon^{1/4}$.

The asymptotic results for laminated glass are presented in Table 1. In the first column the asymptotic orders of the coefficients γ_i in the expansion (11) at $\varepsilon \ll 1$ are displayed. In the third and fourth columns the estimates for each term in (11), expressed in terms of the wave number K, are shown for the fundamental mode and first harmonic, respectively.

For the fundamental mode, we get a two-term expansion corresponding to the classical Kirchhoff plate bending theory. It is

$$\varepsilon^{3/2}G_1\Omega^2 + G_2K^4 = 0, \quad \Omega \ll \varepsilon^{1/2} \tag{13}$$

with

$$G_1 = -\frac{h_0^6}{r_0^3}, \quad G_2 = -\frac{4}{3}\frac{h_0^8(\varkappa_s^2-1)}{r_0^2},$$

where $h_0 = \varepsilon^{1/4}h$ and $r_0 = r$, implying that $K \sim \varepsilon^{3/8}\Omega^{1/2}$.

Similarly to the derivation in [8] (see also [9]) in the vicinity of the first shear resonant frequency ($\Omega_{sh} \sim \varepsilon^{1/8}$, see (9)), we have

$$\varepsilon^{5/4}G_1 + G_3K^2 + \varepsilon G_5\Omega^2 = 0, \quad \Omega_{sh} \leq \Omega \ll \varepsilon^{1/16}, \tag{14}$$

where

$$G_3 = \frac{4h_0^7(\varkappa_s^2-1)}{r_0^3}, \quad G_5 = \frac{h_0^7}{r_0^4},$$

resulting in $K \sim \varepsilon^{1/2}\left(\Omega^2 - \Omega_{sh}^2\right)^{1/2}$.

Table 1: Asymptotic behaviour at $\varepsilon \ll 1$, $h \sim \varepsilon^{-1/4}$, and $r \sim 1$

Order of γ_i	Terms	Fundamental mode $K \sim \varepsilon^{3/8}\Omega^{1/2}$ $\Omega \ll 1$	Harmonic $K \sim \varepsilon^{1/2}\left(\Omega^2 - \Omega_{sh}^2\right)^{1/2}$ $\Omega_{sh} \leq \Omega \ll \varepsilon^{1/16}$
$\gamma_1 \sim \varepsilon^{17/4}$	$\gamma_1\Omega^2$	$\varepsilon^{11/4}K^4$	$\varepsilon^{13/4}(K^2 + \varepsilon^{5/4})$
$\gamma_2 \sim \varepsilon^{11/4}$	$\gamma_2 K^4$	$\varepsilon^{11/4}K^4$	$\varepsilon^{11/4}K^4$
$\gamma_3 \sim \varepsilon^3$	$\gamma_3 K^2\Omega^2$	$\varepsilon^{3/2}K^6$	$\varepsilon^2 K^2(K^2 + \varepsilon^{5/4})$
$\gamma_4 \sim \varepsilon^{3/2}$	$\gamma_4 K^6$	$\varepsilon^{3/2}K^6$	$\varepsilon^{3/2}K^6$
$\gamma_5 \sim \varepsilon^4$	$\gamma_5\Omega^4$	εK^8	$\varepsilon^2(K^2 + \varepsilon^{5/4})^2$
$\gamma_6 \sim \varepsilon^{5/2}$	$\gamma_6 K^4\Omega^2$	εK^8	$\varepsilon^{3/2}K^4(K^2 + \varepsilon^{5/4})$
$\gamma_7 \sim \varepsilon$	$\gamma_7 K^8$	εK^8	εK^8
$\gamma_8 \sim \varepsilon^3$	$\gamma_8\Omega^4 K^2$	K^{10}	$\varepsilon K^2(K^2 + \varepsilon^{5/4})^2$
$\gamma_9 \sim \varepsilon^{3/2}$	$\gamma_9 K^6\Omega^2$	K^{10}	$\varepsilon^{1/2}K^6(K^2 + \varepsilon^{5/4})$
$\gamma_{10} \sim \varepsilon^{1/2}$	$\gamma_{10}K^{10}$	$\varepsilon^{1/2}K^{10}$	$\varepsilon^{1/2}K^{10}$

Further analysis of entries in Table 1 leads to the two-mode uniform approximation

$$\varepsilon^{11/4}G_1\Omega^2 + \varepsilon^{5/4}G_2K^4 + \varepsilon^{3/2}G_3K^2\Omega^2 + G_4K^6 + \varepsilon^{5/2}G_5\Omega^4 = 0, \tag{15}$$

where G_i $(i = 1, 2, 3, 5)$ have been introduced above and

$$G_4 = \frac{4}{3}\frac{h_0^9(\varkappa_s^2 - 1)^2}{r_0^2}.$$

Approximation (15) is uniform in nature since it is valid over the entire low-frequency range $\Omega \ll 1$ for both fundamental mode and first harmonic. Numerical illustration in Figure 2 demonstrates a good agreement between the asymptotic expansion (15) and the Rayleigh-Lamb dispersion relation (A.1). In this figure the parameters of laminated glass are $\varepsilon \approx 0.00002$, $h = 10$, $r = 0.428$, $\varkappa_c \approx 0.302$, and $\varkappa_s \approx 0.592$, see [1]. In this figure a low-frequency approximation is valid over a surprisingly broad frequency range. A similar phenomenon was previously observed in [10] for a homogeneous nearly incompressible plate with fixed faces.

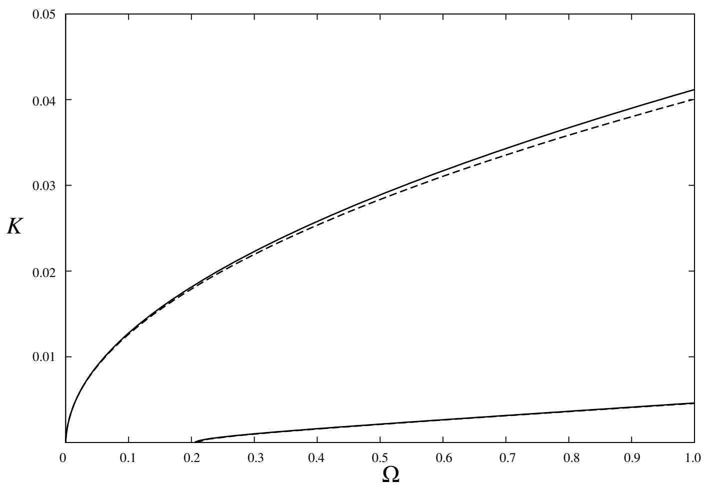

Fig. 2: The dispersion curves for the uniform two-mode approximation (15) (dashed line) and the Rayleigh-Lamb dispersion relation (A.1) (solid line) at $\varepsilon \approx 0.00002$, $h = 10$, $r = 0.428$, $\varkappa_c \approx 0.302$, and $\varkappa_s \approx 0.592$

In what follows we need a more precise expansion for the fundamental mode, since the Kirchhoff plate approximation (13) is not valid at the shear thickness resonance $\Omega_{sh} \sim \varepsilon^{1/8}$. On taking into account the asymptotic results summarized in the third column of Table 1 we arrive at

$$\varepsilon^{11/4}G_1\Omega^2 + \varepsilon^{5/4}G_2K^4 + \varepsilon^{3/2}G_3K^2\Omega^2 + G_4K^6 = 0, \qquad \Omega \ll 1. \tag{16}$$

The latter involves two local limiting behaviours, including (13) and

$$\varepsilon^{3/2}G_3\Omega^2 + G_4K^4 = 0, \quad \varepsilon^{1/2} \ll \Omega \ll 1. \tag{17}$$

In Figure 3 the four-term approximation for the fundamental mode (16) and that, corresponding to Kirchhoff plate theory (13) are displayed together with exact solution of the dispersion relation (A.1) for the same set of parameters as in Figure 2. It is clear that the classical theory is hardly applicable in case of a high contrast. At the same time, (16) is accurate over the whole low-frequency region.

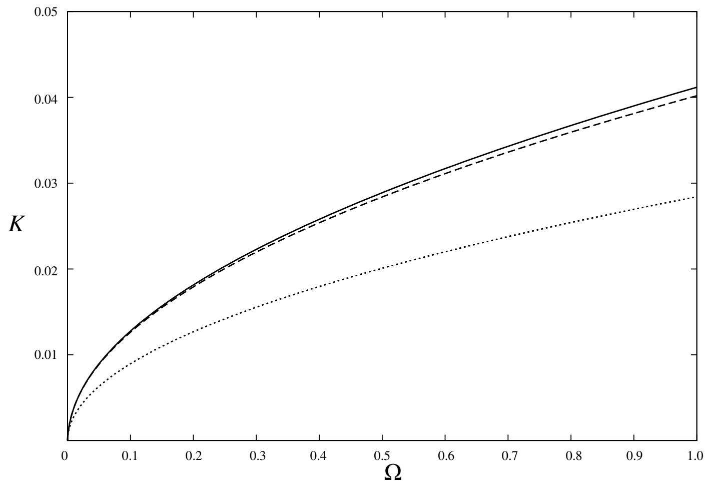

Fig. 3: The four-term approximation for the fundamental mode (16) (dashed line) together with Kirchhoff approximation (13) (dotted line) and the Rayleigh-Lamb dispersion relation (A.1) (solid line) for the same set of parameters as in Figure 2

4. Conclusion

A two-mode uniform asymptotic expansion for the antisymmetric dispersion relation incorporating contrast parameters of laminated glass is derived. Local approximations for both fundamental and lowest shear modes are studied including analysis of their validity range. Numerical results demonstrate a good agreement of the derived polynomial approximations and the transcendental Rayleigh-Lamb dispersion equation. The theory can be further developed, requiring a non-trivial consideration of edge boundary conditions. The obtained results can also be extended to a viscoelastic interlayer embedded into laminated glass. Various boundary conditions, motivated by industrial applications, e.g. see [3], can also be tackled.

Appendix A. Rayleigh-Lamb dispersion relation for antisymmetric motion

$$\begin{aligned}
&4K^2h^3\alpha_s\beta_sF_4\left[F_1F_2C_{\beta_c}S_{\alpha_c} - 2\alpha_c\beta_c(\varepsilon-1)F_3C_{\alpha_c}S_{\beta_c}\right] + \\
&h\alpha_s\beta_sC_{\alpha_s}C_{\beta_s}\left[4\alpha_c\beta_cK^2\left(h^4F_3^2 + {F_4}^2(\varepsilon-1)^2\right)C_{\alpha_c}S_{\beta_c} - \left(4K^4h^4F_2^2 + {F_4}^2F_1^2\right)S_{\alpha_c}C_{\beta_c}\right] + \\
&C_{\beta_s}S_{\alpha_s}\varepsilon\beta_s(\beta_s^2 - K^2h^2)(\beta_c^2 - K^2)\left[4\alpha_s^2\beta_cK^2h^2S_{\alpha_c}S_{\beta_c} - {F_4}^2\alpha_cC_{\alpha_c}C_{\beta_c}\right] + \\
&C_{\alpha_s}S_{\beta_s}\varepsilon\alpha_s(\beta_s^2 - K^2h^2)(\beta_c^2 - K^2)\left[4\alpha_c\beta_s^2K^2h^2C_{\alpha_c}C_{\beta_c} - {F_4}^2\beta_cS_{\alpha_c}S_{\beta_c}\right] + \\
&h^3S_{\alpha_s}S_{\beta_s}\left[\left(4\alpha_s^2\beta_s^2K^2F_1^2 + K^2{F_4}^2F_2^2\right)C_{\beta_c}S_{\alpha_c} - \alpha_c\beta_c\left(16\alpha_s^2\beta_s^2(\varepsilon-1)^2K^4 + {F_4}^2F_3^2\right)C_{\alpha_c}S_{\beta_c}\right] = 0,
\end{aligned} \tag{A.1}$$

where

$$F_1 = 2(\varepsilon - 1)K^2 - \varepsilon\Omega^2, \quad F_2 = 2(\varepsilon - 1)K^2 + \frac{\varepsilon(1-r)}{r}\,\Omega^2, \quad F_3 = 2(\varepsilon - 1)K^2 + \frac{\varepsilon}{r}\,\Omega^2, \quad F_4 = \beta_s^2 + K^2h^2,$$

and

$$\alpha_c^2 = K^2 - \varkappa_c^2\Omega^2, \quad \alpha_s^2 = h^2\left(K^2 - \frac{\varepsilon\varkappa_s^2}{r}\,\Omega^2\right), \quad \beta_c^2 = K^2 - \Omega^2, \quad \beta_s = h^2\left(K^2 - \frac{\varepsilon}{r}\,\Omega^2\right).$$

In the above $C_{\alpha_q} = \cosh(\alpha_q)$, $C_{\beta_q} = \cosh(\beta_q)$, $S_{\alpha_q} = \sinh(\alpha_q)$, $S_{\beta_q} = \sinh(\beta_q)$, and $\varkappa_q = c_{2q}/c_{1q}$ with

$$c_{1q}^2 = \frac{\lambda_q + 2\mu_q}{\rho_q}, \quad c_{2q}^2 = \frac{\mu_q}{\rho_q}, \quad q = c, s.$$

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