PROCEEDINGS A

rspa.royalsocietypublishing.org

Research

Article submitted to journal

Keywords:
nonlocal, elasticity, asymptotic, boundary conditions, Rayleigh wave, inhomogeneous

Author for correspondence:
J. Kaplunov
e-mail: j.kaplunov@keele.ac.uk

THE ROYAL SOCIETY
PUBLISHING

Refined boundary conditions on the free surface of an elastic half-space taking into account nonlocal effects

R. Chebakov[1], J. Kaplunov[2] and G. A. Rogerson[3]

[1,2,3]School of Computing and Mathematics, Keele University, Keele, ST5 5BG, UK

The dynamic response of a homogeneous half-space, with a traction-free surface, is considered within the framework of nonlocal elasticity. The focus is on the dominant effect of the boundary layer on overall behaviour. A typical wavelength is assumed to considerably exceed the associated internal lengthscale. The leading order long-wave approximation is shown to coincide formally with the 'local' problem for a half-space with a vertical inhomogeneity localised near the surface. Subsequent asymptotic analysis of the inhomogeneity results in an explicit correction to the classical boundary conditions on the surface. The order of the correction is greater than the order of the better-known correction to the governing differential equations. The refined boundary conditions enable us to evaluate the interior solution outside a narrow boundary layer localised near the surface. As an illustration, the effect of nonlocal elastic phenomena on the Rayleigh wave speed is investigated.

© The Authors. Published by the Royal Society under the terms of the Creative Commons Attribution License http://creativecommons.org/licenses/by/4.0/, which permits unrestricted use, provided the original author and source are credited.

1. Introduction

Analysis of nonlocal elastic phenomena is of major interest for various advanced applications including micro- and nanomechanics, see for example Drugan and Willis (1996), Arash and Wang (2012), Dal Corso and Deseri (2013). Nonlocal elasticity is a particularly powerful and appropriate theory for investigating properties of solids with impurities, dislocations, and granular microstructure. The fundamental concepts underpinning contemporary nonlocal continuum models were developed in a series of well known papers by Kroner (1967), Eringen (1972), Eringen and Edelen (1972); see also Eringen (2002) and references therein. The state of art has been presented by a number of authors throughout the area's scientific development, see Krumhansl (1968), Kunin (1984), Polizzotto (2001), Peddieson et al. (2003), Di Paola et al. (2013), and Dell'Isola et al. (2014). The latter paper addresses important Piola's contribution, not widely known for a long time to a broad international audience, see also references to original Piola's papers in Dell'Isola et al. (2014).

Among other recent publications on the subject, we mention papers by Di Paola and Zingales (2008), Di Paola et al. (2009), Zingales (2011), Schwartz et al. (2012), Benvenuti and Simone (2013), Abdollahi and Boroomand (2013), and Abdollahi and Boroomand (2014), dealing with various analytical and numerical aspects of nonlocal elasticity. Here we also cite publications developing novel micromechanical approaches known as 'structured deformations', e.g., see Owen and Paroni (2000), Del Piero and Owen (2004), and Owen and Paroni (2015).

Nonlocal models, e.g., see Eringen (1966), Eringen (1983), and Eringen (1987), are oriented to the investigation of the distant interaction between small material particles, assuming that the stress at a reference point is dependent upon the entire strain field in the body. The associated constitutive relations are usually expressed through integral operators involving internal sizes which characterise microstructure. As a rule, e.g., see Rogula (1982), the long-wave limit of the nonlocal elasticity relations is identical to its classical counterpart. We also remark that a number of nonlocal elasticity predictions are in good agreement with lattice dynamics, including the regions near the boundaries of the body (Eringen and Kim, 1977).

In spite of the numerous publications, the fundamental effect of boundaries on the implementation of nonlocal elasticity concepts has not yet been properly addressed. The key point is that the intervals of integration corresponding to the above mentioned operators, expressing nonlocal constitutive relations, are dependent of the distance from a reference point to the boundary (Eringen, 1983). This results in boundary layers corresponding to localised nonhomogeneous stress and strain fields. In the present paper, we fill the gap in tackling the influence of boundary layers on overall dynamic behaviour. Although several authors emphasised the crucial role of boundary layers, e.g., see Bazant et al. (2010) and Abdollahi and Boroomand (2014), we are not aware of any related asymptotic developments.

As an example, we consider an elastic half-space governed by the nonlocal equations given in Eringen (1983), see Section 2. For the sake of definiteness, we assume that the nonlocal behaviour is modelled by an exponential kernel involving a small internal lengthscale. In Section 3, we proceed with a long-wave asymptotic scheme, originating from Goldenveizer et al. (1993) and later developed by, for example, Dai et al. (2010) and Aghalovyan (2014). Within the framework of these studies, the characteristic wavelength is assumed to be much greater than a typical microscale parameter. We begin by reducing the original nonlocal problem to a formulation which is identical to the classical problem for an elastic half-space with a vertical inhomogeneity localised near the surface. The effect of the inhomogeneity can be reduced to effective boundary conditions imposed at a near-surface interface. In this case, we can only asymptotically evaluate the interval, yielding the location of the interface. A better option seems to be a transformation of the effective conditions to refined boundary conditions along the surface of a homogeneous half-space. This approach is exploited in Section 4, enabling us to evaluate the interior stress and strain outside the narrow boundary layer. In Section 5, the refined boundary conditions are applied to calculate the nonlocal correction to the Rayleigh surface wave. The order of this correction exceeds that of the correction established in Eringen (1983) associated with the nonlocal differential equations of motion.

2. Equations of nonlocal linear elasticity

In this section, we use as our starting point the equations of nonlocal elasticity, e.g., see Eringen (1983). For a homogeneous isotropic elastic solid, we therefore have (2.1)-(2.5) below:

$$s_{\alpha\beta,\alpha} = \rho \frac{\partial^2 u_\beta}{\partial t^2}\,, \tag{2.1}$$

with $u_\beta, \beta = 1, 2, 3$, the components of the displacement vector, ρ volume density, t time and

$$s_{\alpha\beta}(\mathbf{x}) = \int\limits_V K\left(|\mathbf{x}' - \mathbf{x}|, a\right) \sigma_{\alpha\beta}(\mathbf{x}')\, dv(\mathbf{x}')\,, \tag{2.2}$$

where $s_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ are the nonlocal and classical stress tensors, respectively, considered at time t, $\mathbf{x} = (x_1, x_2, x_3)$ is a reference point, V the domain occupied by the body, $K(\mathbf{x}, a)$ the so-called nonlocal modulus, and a is an internal characteristic length, e.g. lattice parameter or granular distance. Throughout the paper we assume that the internal size a is asymptotically small in comparison with a typical wavelength. This long-wave assumption provides the validity of the adapted nonlocal model for bounded domains as it follows on, in particular, from lattice dynamics (Eringen and Kim, 1977); for further details, see concluding remarks.

The function K in (2.2) is normalised over 3D space, so that

$$\int\limits_{V_\infty} K\left(|\mathbf{x}'|, a\right)\; dv(\mathbf{x}') = 1\,. \tag{2.3}$$

The two equations (2.1) and (2.2) are accompanied by

$$\sigma_{\alpha\beta} = \lambda e_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta} \tag{2.4}$$

and

$$e_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)\,, \tag{2.5}$$

where $e_{\alpha\beta}$ is the linear elastic strain tensor, $\delta_{\alpha\beta}$ the Kronecker's delta, and λ and μ are the Lamé constants.

For the sake of definiteness, we specify the 3D exponential nonlocal modulus in the same way as Eringen (1983), thus

$$K(|\mathbf{x}|, a) = \frac{1}{\pi^{3/2} a^3} \exp\left[-\frac{\mathbf{x}\cdot\mathbf{x}}{a^2}\right]\,, \tag{2.6}$$

where in the case of a half-space $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, and $0 \le x_3 < \infty$, (2.2) becomes

$$s_{\alpha\beta}(\mathbf{x}) = \frac{1}{\pi^{3/2} a^3} \int\limits_0^\infty dx_3' \int\limits_{-\infty}^\infty dx_1' \int\limits_{-\infty}^\infty dx_2' \,\exp\left[-\frac{(\mathbf{x}' - \mathbf{x})^2}{a^2}\right] \sigma_{\alpha\beta}(\mathbf{x}')\,. \tag{2.7}$$

Let us now expand the stresses $\sigma_{\alpha\beta}$ in Taylor series about the reference point $\mathbf{x}' = \mathbf{x}$, assuming as before that the typical wavelength characterising the classical stress field is much greater than the internal size a. Thus, we establish from (2.7) that

$$\begin{aligned} s_{\alpha\beta}(\mathbf{x}) &= \frac{1}{a\sqrt{\pi}} \left\{ \sigma_{\alpha\beta}(\mathbf{x}) \int\limits_0^\infty \exp\left[-\frac{(x_3' - x_3)^2}{a^2}\right] dx_3' \right. \\ &\left. + \frac{\partial \sigma_{\alpha\beta}(\mathbf{x})}{\partial x_3} \int\limits_0^\infty (x_3' - x_3) \exp\left[-\frac{(x_3' - x_3)^2}{a^2}\right] dx_3' \right\} \\ &+ \dots\,, \end{aligned} \tag{2.8}$$

which on integration yields

$$s_{\alpha\beta}(\mathbf{x}) = \frac{\sigma_{\alpha\beta}(\mathbf{x})}{2}\,\mathrm{erfc}\left(-\frac{x_3}{a}\right) + \frac{a}{2\sqrt{\pi}}\frac{\partial\sigma_{\alpha\beta}(\mathbf{x})}{\partial x_3}\exp\left[-\frac{x_3^2}{a^2}\right] + \dots\,, \tag{2.9}$$

where $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int\limits_x^\infty e^{-t^2}dt$. Here we keep only a linear term in a which is specific for a half-space. Such term does not appear in the case of 3D space.

The formulae (2.2), taking into account (2.4) and keeping the leading order term in (2.9), may be presented as

$$s_{\alpha\beta} = \lambda' e_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu' e_{\alpha\beta}\,, \tag{2.10}$$

where

$$\begin{aligned}\lambda' &= \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x_3}{a}\right)\lambda\,,\\ \mu' &= \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x_3}{a}\right)\mu\,.\end{aligned} \tag{2.11}$$

We remark that the nonlocal problem (2.1),(2.10)-(2.11) is thus formally equivalent to the classical ('local') problem for a vertically inhomogeneous elastic half-space. This problem may in fact be reduced to that of analysis of a homogeneous elastic substrate coated by a vertically inhomogeneous layer of a certain thickness h, see Figure 1, where $h \gg a$. This strong inequality justifies the validity of nonlocal theory on the scale of layer thickness. As a rule, the asymptotic error of the one-term expansion in (2.9) is $O(\frac{a}{h})$. It is less than $O(\frac{a}{h})$ only provided that the associated local field is uniform in x_3.

Along the interface $x_3 = h$, to within an exponentially small error, $\mathrm{erfc}\left(-\frac{h}{a}\right) = 2$ and, consequently,

$$\lambda'(h) = \lambda \quad \text{and} \quad \mu'(h) = \mu \tag{2.12}$$

and the nonlocal stresses $s_{\alpha\beta}$ tend to their local analogues $\sigma_{\alpha\beta}$ in (2.9).

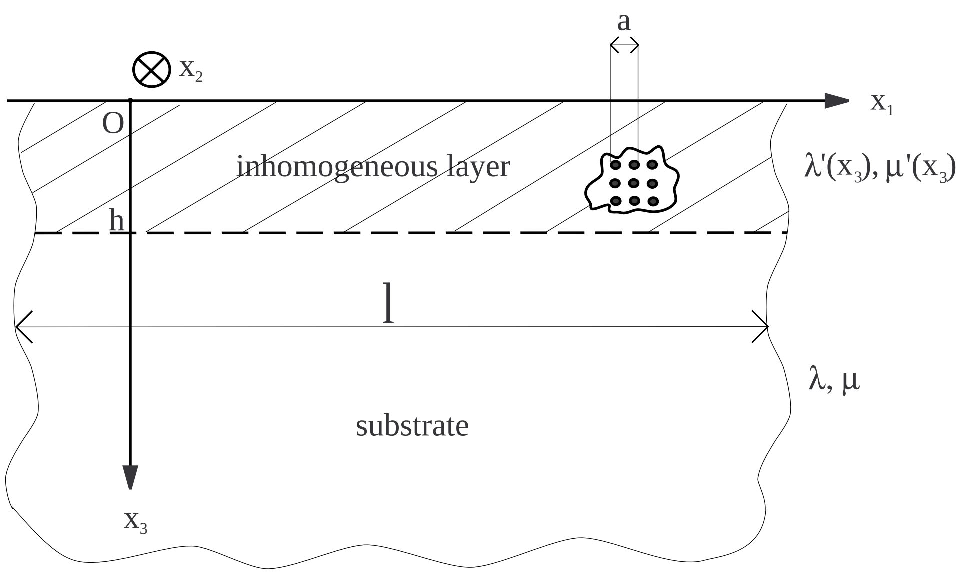

Figure 1. A homogeneous substrate coated by a vertically inhomogeneous layer of thickness h; $a \ll h \ll \ell$.

In the case of a thin layer of thickness much smaller than a macroscale wavelength, its effect on the substrate may be incorporated by deriving effective boundary conditions using well known asymptotic methodology, see for example Dai et al. (2010) and Aghalovyan (2014), and references therein.

3. Asymptotic analysis of a vertically inhomogeneous thin layer

Let us consider a thin, vertically inhomogeneous layer of thickness $h \ll \ell$, where ℓ is a typical wavelength with $\varepsilon = \frac{h}{\ell}$ assumed to be a small geometric parameter. Equations (2.1) and (2.10) in the previous section are formally identical to the classical 'local' equations. They can be rewritten as

$$\begin{aligned} \frac{\partial s_{ii}}{\partial x_i} + \frac{\partial s_{ij}}{\partial x_j} + \frac{\partial s_{3i}}{\partial x_3} = \rho \frac{\partial^2 u_i}{\partial t^2} \, , \\ \frac{\partial s_{3i}}{\partial x_i} + \frac{\partial s_{3j}}{\partial x_j} + \frac{\partial s_{33}}{\partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2} \, , \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} & s_{ij} = \rho c_2'^2(x_3) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \, , \\ & s_{ii} = \rho c_1'^2(x_3) \frac{\partial u_i}{\partial x_i} + \rho (c_1'^2(x_3) - 2 c_2'^2(x_3)) \left(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_3}{\partial x_3} \right) \, , \\ & s_{3i} = \sigma_{i3} = \rho c_2'^2(x_3) \left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i} \right) \, , \\ & s_{33} = \rho c_1'^2(x_3) \frac{\partial u_3}{\partial x_3} + \rho (c_1'^2(x_3) - 2 c_2'^2(x_3)) \left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j} \right) \, , \end{aligned} \tag{3.2}$$

where $i \neq j = 1, 2$ and Einstein's summation convention is not employed. The variable wave speeds in (3.2), inspired by (2.11), are given by

$$c_1'(x_3) = \sqrt{\frac{\lambda'(x_3) + 2\mu'(x_3)}{\rho}} \quad \text{and} \quad c_2'(x_3) = \sqrt{\frac{\mu'(x_3)}{\rho}} \, . \tag{3.3}$$

The traction-free boundary conditions at the surface of the layer $x_3 = 0$ are given by

$$s_{3n} = 0 \quad \text{at } x_3 = 0 \, , \tag{3.4}$$

with continuity of displacement along the interface $x_3 = h$ requiring that

$$u_n = v_n \quad \text{at } x_3 = h \, , \tag{3.5}$$

where $v_n = v_n(x_1, x_2, t)$ denotes the prescribed displacements in the substrate, $n = 1, 2, 3$.

We now adapt the asymptotic approach developed in Goldenveizer et al. (1993), Dai et al. (2010), and Aghalovyan (2014) in order to express the stresses s_{3n} along the interface $x_3 = h$ in terms of the prescribed substrate displacements v_n. To begin, we scale the original variables as follows

$$\xi_i = \frac{x_i}{\ell} \, , \quad \eta = \frac{x_3}{h} \, , \quad \text{and} \quad \tau = \frac{t c_2}{\ell} \, , \tag{3.6}$$

where $c_2 = c_2'(h)$, and also define the dimensionless quantities

$$u_n^* = \frac{1}{V} u_n \, , \; v_n^* = \frac{1}{V} v_n$$

and

$$s_{ij}^* = \frac{\ell}{\mu V} s_{ij} \, , \; s_{ii}^* = \frac{\ell}{\mu V} s_{ii} \, , \; s_{3n}^* = \frac{\ell^2}{\mu h V} s_{3n} \, , \tag{3.7}$$

where V is the maximum displacement amplitude and all quantities with an asterisk are assumed to be of the same asymptotic order.

The equations of motion (3.1) and constitutive relations (3.2) can now be rewritten as

$$\begin{aligned} &\frac{\partial s^*_{ii}}{\partial \xi_i} + \frac{\partial s^*_{ij}}{\partial \xi_j} + \frac{\partial s^*_{3i}}{\partial \eta} = \frac{\partial^2 u^*_i}{\partial \tau^2}\,, \\ &\frac{\partial s^*_{33}}{\partial \eta} + \varepsilon\left(\frac{\partial s^*_{3i}}{\partial \xi_i} + \frac{\partial s^*_{3j}}{\partial \xi_j}\right) = \frac{\partial^2 u^*_3}{\partial \tau^2}\,, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &s^*_{ij} = \kappa_2'^2\left(\frac{\partial u^*_i}{\partial \xi_j} + \frac{\partial u^*_j}{\partial \xi_i}\right), \\ &\varepsilon s^*_{ii} = (\kappa_1'^2 - 2\kappa_2'^2)\frac{\partial u^*_3}{\partial \eta} + \varepsilon\left(\kappa_1'^2\frac{\partial u^*_i}{\partial \xi_i} + (\kappa_1'^2 - 2\kappa_2'^2)\frac{\partial u^*_j}{\partial \xi_j}\right), \\ &\varepsilon^2 s^*_{3i} = \kappa_2'^2\left(\frac{\partial u^*_i}{\partial \eta} + \varepsilon\frac{\partial u^*_3}{\partial \xi_i}\right), \\ &\varepsilon^2 s^*_{33} = \kappa_1'^2\frac{\partial u^*_3}{\partial \eta} + \varepsilon(\kappa_1'^2 - 2\kappa_2'^2)\left(\frac{\partial u^*_i}{\partial \xi_i} + \frac{\partial u^*_j}{\partial \xi_j}\right), \end{aligned} \tag{3.9}$$

with $\kappa_m' = \frac{c_m'(x_3)}{c_2}$, $m = 1, 2$.

It is convenient to express $\frac{\partial u^*_3}{\partial \eta}$ in (3.9)$_2$ from (3.9)$_4$, having

$$s^*_{ii} = 4\kappa_2'^2\left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right)\frac{\partial u^*_i}{\partial \xi_i} + 2\kappa_2'^2\left(1 - \frac{2\kappa_2'^2}{\kappa_1'^2}\right)\frac{\partial u^*_j}{\partial \xi_j} + \varepsilon\left(1 - \frac{2\kappa_2'^2}{\kappa_1'^2}\right)s^*_{33}\,. \tag{3.10}$$

The boundary conditions (3.4) and (3.5) become

$$\begin{aligned} &s^*_{3n} = 0 \quad \text{at } \eta = 0\,, \\ &u^*_n = v^*_n \quad \text{at } \eta = 1\,. \end{aligned} \tag{3.11}$$

Next, we expand the displacements and stresses in asymptotic series in terms of the previously specified small parameter ε, and thus introduce

$$\begin{pmatrix} u^*_n \\ s^*_{ii} \\ s^*_{ij} \\ s^*_{3i} \\ s^*_{33} \end{pmatrix} = \begin{pmatrix} u^{(0)}_n \\ s^{(0)}_{ii} \\ s^{(0)}_{ij} \\ s^{(0)}_{3i} \\ s^{(0)}_{33} \end{pmatrix} + \varepsilon\begin{pmatrix} u^{(1)}_n \\ s^{(1)}_{ii} \\ s^{(1)}_{ij} \\ s^{(1)}_{3i} \\ s^{(1)}_{33} \end{pmatrix} + \dots \tag{3.12}$$

Substitution of these expressions into equations (3.8) - (3.10), and boundary conditions (3.11) results, at leading order, in the following equations

$$\begin{aligned} &\frac{\partial s^{(0)}_{ii}}{\partial \xi_i} + \frac{\partial s^{(0)}_{ij}}{\partial \xi_j} + \frac{\partial s^{(0)}_{3i}}{\partial \eta} = \frac{\partial^2 u^{(0)}_i}{\partial \tau^2}\,, \\ &\frac{\partial s^{(0)}_{33}}{\partial \eta} = \frac{\partial^2 u^{(0)}_3}{\partial \tau^2}\,, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &s^{(0)}_{ij} = \kappa_2'^2\left(\frac{\partial u^{(0)}_i}{\partial \xi_j} + \frac{\partial u^{(0)}_j}{\partial \xi_i}\right), \\ &s^{(0)}_{ii} = 4\kappa_2'^2\left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right)\frac{\partial u^{(0)}_i}{\partial \xi_i} + 2\kappa_2'^2\left(1 - \frac{2\kappa_2'^2}{\kappa_1'^2}\right)\frac{\partial u^{(0)}_j}{\partial \xi_j}\,, \\ &\frac{\partial u^{(0)}_n}{\partial \eta} = 0\,, \end{aligned} \tag{3.14}$$

together with

$$\begin{aligned} &u^{(0)}_n = v^*_n \quad \text{at } \eta = 1\,, \\ &s^{(0)}_{3n} = 0 \quad \text{at } \eta = 0\,. \end{aligned} \tag{3.15}$$

On integrating (3.13)$_2$ and (3.14)$_3$ with respect to η, and taking into account the appropriate boundary conditions (3.15), we may establish that

$$u_n^{(0)} = v_n^* \tag{3.16}$$

and

$$s_{33}^{(0)} = \eta \frac{\partial^2 v_3^*}{\partial \tau^2}\,. \tag{3.17}$$

Now we obtain from (3.14)$_2$

$$s_{ii}^{(0)} = 4\kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) \frac{\partial v_i^*}{\partial \xi_i} + 2\kappa_2'^2 \left(1 - \frac{2\kappa_2'^2}{\kappa_1'^2}\right) \frac{\partial v_j^*}{\partial \xi_j}\,. \tag{3.18}$$

We finally integrate (3.13)$_1$, using (3.14)$_2$ and (3.16), and then satisfy (3.15)$_2$, to establish that

$$\begin{aligned} s_{3i}^{(0)} = \eta \frac{\partial^2 v_i^*}{\partial \tau^2} - \frac{\partial^2 v_i^*}{\partial \xi_j^2} \int_0^\eta \kappa_2'^2\, d\eta' - 4 \frac{\partial^2 v_i^*}{\partial \xi_i^2} \int_0^\eta \kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) d\eta' \\ - \frac{\partial^2 v_j^*}{\partial \xi_i \partial \xi_j} \int_0^\eta \kappa_2'^2 \left(3 - \frac{4\kappa_2'^2}{\kappa_1'^2}\right) d\eta'\,. \end{aligned} \tag{3.19}$$

In terms of the original variables, the expressions for the stresses s_{3i} and s_{33} may be obtained from (3.19) and (3.17) in the form

$$\begin{aligned} s_{3i} &= \rho \left[x_3 \frac{\partial^2 u_i}{\partial t^2} - c_2^2 \frac{\partial^2 u_i}{\partial x_j^2} \int_0^{x_3} \kappa_2'^2\, dx_3' - \rho c_2^2 \frac{\partial^2 u_i}{\partial x_i^2} \int_0^{x_3} 4\kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) dx_3' \right. \\ &\qquad \left. - c_2^2 \frac{\partial^2 u_j}{\partial x_i \partial x_j} \int_0^{x_3} \kappa_2'^2 \left(3 - \frac{4\kappa_2'^2}{\kappa_1'^2}\right) dx_3' \right], \\ s_{33} &= \rho x_3 \frac{\partial^2 u_3}{\partial t^2}\,, \end{aligned} \tag{3.20}$$

where now $u_n = V u_n^{(0)}$.

In what follows we also use the formula for other components of the nonlocal stress tensor, which are given by

$$\begin{aligned} s_{ii} &= 2\rho c_2'^2 \left[2\left(1 - \frac{c_2'^2}{c_1'^2}\right) \frac{\partial u_i}{\partial x_i} + \left(1 - \frac{2c_2'^2}{c_1'^2}\right) \frac{\partial u_j}{\partial x_j} \right], \\ s_{ij} &= \rho c_2'^2 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \end{aligned} \tag{3.21}$$

The stresses at the interface, $x_3 = h$, may be expressed through the substrate displacements, yielding

$$\begin{aligned} s_{3i} &= \rho \left[h \frac{\partial^2 v_i}{\partial t^2} - c_2^2 \frac{\partial^2 v_i}{\partial x_j^2} \int_0^{h} \kappa_2'^2\, dx_3' - \rho c_2^2 \frac{\partial^2 v_i}{\partial x_i^2} \int_0^{h} 4\kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) dx_3' \right. \\ &\qquad \left. - c_2^2 \frac{\partial^2 v_j}{\partial x_i \partial x_j} \int_0^{h} \kappa_2'^2 \left(3 - \frac{4\kappa_2'^2}{\kappa_1'^2}\right) dx_3' \right], \\ s_{33} &= \rho h \frac{\partial^2 v_3}{\partial t^2}\,, \end{aligned} \tag{3.22}$$

where $\kappa_m' = \frac{c_m'}{c_2}$, $m = 1, 2$ as above.

4. Refined boundary conditions

For a vertically inhomogeneous layer with the elastic moduli given by (2.11), we obtain

$$c_m'^2(x_3) = \frac{1}{2} c_m^2 \operatorname{erfc}\left(-\frac{x_3}{a}\right), m = 1, 2, \tag{4.1}$$

where, see (2.12),

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_2 = \sqrt{\frac{\mu}{\rho}}.$$

Consequently, the integrals in (3.22), under the assumption $a \ll h$, to within an exponentially small error may be presented in the forms

$$\int_0^h \kappa_2'^2 \, dx_3' = h\left(1 - \frac{1}{2\sqrt{\pi}} \frac{a}{h}\right),$$

$$\int_0^h 4\kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) dx_3' = 4h(1-\kappa^2)\left(1 - \frac{1}{2\sqrt{\pi}} \frac{a}{h}\right),$$

and

$$\int_0^h \kappa_2'^2 \left(3 - \frac{4\kappa_2'^2}{\kappa_1'^2}\right) dx_3' = h(3-4\kappa^2)\left(1 - \frac{1}{2\sqrt{\pi}} \frac{a}{h}\right),$$

where $\kappa = \frac{c_2}{c_1}$. Thus, the stresses along the interface $x_3 = h$ may be presented as

$$\begin{aligned} s_{3i} = \quad & \rho h \left[\frac{\partial^2 v_i}{\partial t^2} - c_2^2 \left\{\frac{\partial^2 v_i}{\partial x_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i \partial x_j}\right\}\right. \\ & \left. + c_2^2 \frac{a}{2h\sqrt{\pi}} \left\{\frac{\partial^2 v_i}{\partial x_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i \partial x_j}\right\}\right], \\ s_{33} = \quad & \rho h \frac{\partial^2 v_3}{\partial t^2}. \end{aligned} \tag{4.2}$$

Inspection of $(4.2)_1$ shows that taking nonlocal elastic properties into account results in an asymptotic correction of the relative asymptotic order $O(\frac{a}{h})$. On the other hand, this correction must be greater than the truncation error $O(\varepsilon)$ related to the asymptotic derivation of formulae (3.22) in Section 3. Thus, we arrive at the double strong inequality

$$a \ll h \ll \sqrt{a\ell}, \tag{4.3}$$

underlying equations (4.2). We also need to show that the accuracy of the leading order approximation (2.10) is consistent with the $O(\frac{a}{h})$ correction associated with (4.2). To this end, we recall that at leading order the stresses s_{3i} and s_{33} are expressed in terms of the stresses s_{ii} and the displacements u_n in Section 3, see (3.13). In this case, the local stresses σ_{ii} and σ_{ij} corresponding to their nonlocal counterparts s_{ii} and s_{ij} in formula (3.21), following from the dimensionless formula $(3.13)_1$, are given by

$$\begin{aligned} \sigma_{ii} &= 2\rho c_2^2 \left[2(1-\kappa^2)\frac{\partial u_i}{\partial x_i} + (1-2\kappa^2)\frac{\partial u_j}{\partial x_j}\right], \\ \sigma_{ij} &= \rho c_2^2 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \end{aligned} \tag{4.4}$$

These stresses are uniform across the thickness. As a result, the expected $O(\frac{a}{h})$ contribution of the second term in the expansion (2.9) vanishes after differentiation with respect to x_3. For the same reason, the inertial terms in (3.13) will also not make a $O(\frac{a}{h})$ contribution to nonlocal stresses.

It is clear that outside a narrow near-surface layer all nonlocal stresses, to within the asymptotic error less than $O(\frac{a}{h})$, coincide with their local analogues, see (2.9) - (2.12). Therefore, over the interior domain $x_3 \geq h$, we may proceed with a classical problem with constant coefficients, subject to the boundary conditions

$$\sigma_{3n} = s_{3n} \quad \text{at} \quad x_3 = h\,, \tag{4.5}$$

where s_{3n} are given by (4.2).

We are now in a position to formulate an inverse problem for a thin homogeneous elastic layer within the classical framework. A crucial aspect that now needs addressing concerns the boundary conditions to be imposed on the surface $x_3 = 0$ so that the stresses σ_{3i} and σ_{33}, at the interface $x_3 = h$, satisfy the conditions (4.5). Let the boundary conditions at the surface of the layer $x_3 = 0$ be given by

$$\sigma_{3n} = p_n\,, \tag{4.6}$$

where p_n are the sought for surface stresses.

The asymptotic solution of the classical elastodynamic equations for a thin homogeneous layer, subject to the boundary conditions (4.5) and (4.6) along the faces $x_3 = 0$ and $x_3 = h$, is presented in Dai et al. (2010). The formulae for the stresses of interest at $x_3 = h$ may be written as

$$\begin{aligned}\sigma_{3i} = &\ \rho h \left[\frac{\partial^2 v_i}{\partial t^2} - c_2^2 \left\{ \frac{\partial^2 v_i}{\partial x_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i \partial x_j} \right\} \right] + p_i\,,\\ \sigma_{33} = &\ \rho h \frac{\partial^2 v_3}{\partial t^2} + p_3\,.\end{aligned} \tag{4.7}$$

Next, on equating the stresses σ_{3n} in (4.5) and (4.7), we have

$$\begin{aligned}&p_i = \rho c_2^2 \frac{a}{2\sqrt{\pi}} \left\{ \frac{\partial^2 v_i}{\partial x_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i \partial x_j} \right\}\,,\\ &p_3 = 0\,.\end{aligned} \tag{4.8}$$

Thus, the refined boundary conditions, at $x_3 = 0$, become

$$\begin{aligned}&\sigma_{3i} = \rho c_2^2 \frac{a}{2\sqrt{\pi}} \left\{ \frac{\partial^2 v_i}{\partial x_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i \partial x_j} \right\}\,,\\ &\sigma_{33} = 0\,.\end{aligned} \tag{4.9}$$

Outside a narrow boundary layer ($x_3 \gg a$), the half-space motion is governed by the elastodynamic equations with constant moduli λ and μ, subject to the boundary conditions (4.9). The last formulae involve $O(\frac{a}{\ell})$ correction, where ℓ is a typical macroscale size as described above. This is greater than the $O(\frac{a^2}{\ell^2})$ correction in the differential equations of nonlocal elasticity, see Eringen (1983).

5. Rayleigh surface wave

As an illustration, we consider the effect of nonlocal elastic behaviour on surface wave propagation in the case of plane strain, in which $\frac{\partial}{\partial x_2} \equiv 0$, $u_m = u_m(x_1, x_3)$, $m = 1, 3$, and $u_2 = 0$. Accordingly, the two boundary conditions, following directly from (4.9), become

$$\begin{aligned}&\sigma_{31} = \frac{2a}{\sqrt{\pi}} \rho c_2^2 (1-\kappa^2) \frac{\partial^2 u_1}{\partial x_1^2}\,,\\ &\sigma_{33} = 0\,,\end{aligned} \tag{5.1}$$

and the equations of motion, in terms of wave potentials φ and ψ, are given by

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = 0\,, \tag{5.2}$$

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0\,, \tag{5.3}$$

where Δ is the 2D Laplacian.

We first look for travelling wave solutions of the form

$$\begin{aligned}\varphi &= Ae^{-rx_3+ik(x_1-ct)}\,,\\ \psi &= Be^{-qx_3+ik(x_1-ct)}\,,\end{aligned} \tag{5.4}$$

where c is the phase speed and in which the attenuation coefficients are given by

$$r = k\sqrt{1-\frac{c^2}{c_1^2}} \quad \text{and} \quad q = k\sqrt{1-\frac{c^2}{c_2^2}}.$$

The displacements may now be expressed in terms of potentials, thus yielding

$$\begin{aligned}u_1 &= \varphi_{,1} + \psi_{,3} = \left(ikAe^{-rx_3} - qBe^{-qx_3}\right)e^{ik(x_1-ct)}\,,\\ u_3 &= \varphi_{,3} - \psi_{,1} = \left(-rAe^{-rx_3} - ikBe^{-qx_3}\right)e^{ik(x_1-ct)}\,.\end{aligned} \tag{5.5}$$

Next, on substituting (5.5) into the boundary conditions (5.1), we obtain, after taking into account the plane strain forms of (2.4) and (2.5), that

$$\begin{aligned}&\left[2-\frac{c^2}{c_2^2}\right]A + \left[2i\sqrt{1-\frac{c^2}{c_2^2}}\right]B = 0\,,\\ &\left[-i\left(\frac{2ak}{\sqrt{\pi}}(\kappa^2-1) + 2\sqrt{1-\frac{c^2}{c_1^2}}\right)\right]A+\\ &+\left[\left(2-\frac{c^2}{c_2^2}\right) + \frac{2ak}{\sqrt{\pi}}(\kappa^2-1)\sqrt{1-\frac{c^2}{c_2^2}}\right]B = 0\,.\end{aligned} \tag{5.6}$$

The condition for existence of a non-trivial solution of (5.6) yields

$$R(\gamma) - 4\sqrt{\pi}\theta(\kappa^2-1)\gamma^2\sqrt{1-\gamma^2} = 0\,, \tag{5.7}$$

where $\theta = \frac{a}{\ell} = \frac{ak}{2\pi} \ll 1$ is a small parameter, $\gamma = \frac{c}{c_2}$ and $R(\gamma)$ is the Rayleigh denominator, i.e.

$$R(\gamma) = (2-\gamma^2)^2 - 4\sqrt{(1-\gamma^2)(1-\kappa^2\gamma^2)}\,.$$

We may now expand γ as an asymptotic series in the small parameter θ, with

$$\gamma = \gamma_0 + \theta\gamma_1 + \dots\,. \tag{5.8}$$

In this case, the Taylor series expansion of $R(\gamma)$, about $\gamma = \gamma_0$, is given by

$$R(\gamma) = R(\gamma_0) + R'(\gamma_0)(\gamma-\gamma_0) + \dots\,, \tag{5.9}$$

where γ_0 is the normalised classical Rayleigh wave speed, i.e. $R(\gamma_0) = 0$. Then, on substituting (5.8) and (5.9) into (5.7), we readily obtain

$$\gamma_1 = \frac{4\sqrt{\pi}(\kappa^2-1)\gamma_0^2\sqrt{1-\gamma_0^2}}{R'(\gamma_0)} \tag{5.10}$$

and, consequently,

$$\gamma = \gamma_0 + \theta\frac{4\sqrt{\pi}(\kappa^2-1)\gamma_0^2\sqrt{1-\gamma_0^2}}{R'(\gamma_0)} + \dots\,, \tag{5.11}$$

where $\theta = \frac{ak}{2\pi}$, as before.

We remark that the constructed correction, originating from the refined boundary conditions (5.1), exceeds the correction in Eringen (1983), associated with the 'nonlocal terms' within the differential equations of motion.

Numerical results are presented in Figure 2. The classical Rayleigh root γ_0 and the 'nonlocal' root γ in (5.11) are plotted as function of the small parameter a for the value of Poisson ratio $\nu = 0.25$. For this scenario, the coefficient (5.10) takes the value $\gamma_1 = -0.37$, while its 'local' counterpart is $\gamma_0 = 0.92$. The effect of nonlocal phenomena decreases the Rayleigh wave speed due to low values of the Lamé parameters, denoting the stiffness of the system, near the surface, see (2.11).

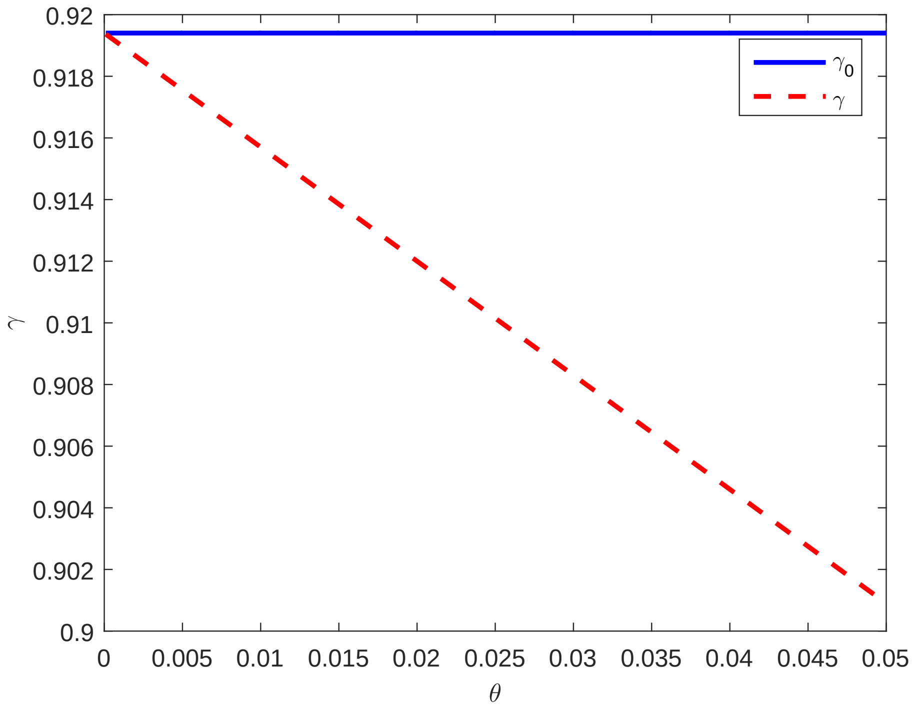

Figure 2. Effect of nonlocal phenomena on Rayleigh wave speed.

6. Concluding remarks

An asymptotic treatment of the nonlocal boundary value problem under consideration demonstrates the primary importance of analysing the peculiarities of near-surface behaviour. It has been established that the effect of the associated boundary layer may be incorporated just by refining the boundary conditions in classical elasticity. In particular, the refined boundary conditions (4.9) involve an explicit correction to their classical counterparts; this arises by taking into account nonlocal phenomena.

The linear elastodynamic equations, subject to the derived boundary conditions on the free surface of a homogeneous half-space, enable us to determine the interior stress and strain fields outside a narrow near-surface layer, with thickness satisfying the asymptotic inequality (4.3). As an illustration, $O(\frac{a}{\ell})$ nonlocal correction to the Rayleigh surface wave speed was calculated. This correction is greater than $O(\frac{a^2}{\ell^2})$ correction associated with the nonlocal equations of motion in Eringen (1983).

We recall that approximate nature of nonlocal models originates from truncation of homogenisation procedures, including asymptotic homogenisation for periodic structures, e.g. see Sanchez-Palencia (1980), Panasenko and Bakhvalov (1989), underlying the associated macroscale relations. In this case, the truncation error for the classical boundary conditions should be of the same order as the deviation from the uniform microscale variation of the sought for solution. The latter might be expected to be negligible in comparison with $O(\frac{a}{\ell})$ correction suggested in the paper. In particular, it is $O(\frac{a^2}{\ell^2})$ for a range of periodic lattices (Craster et al., 2010). This issue certainly merits a thorough consideration.

We remark that the proposed approach is not merely restricted to the exponential kernel (2.6) studied in this paper. We envisage similar nonlocal effects for a range of kernels having the same asymptotic behaviour at small internal scales. The results obtained may also readily be extended to nonlocally elastic solids with a boundary of arbitrary shape. Investigation of elastic waveguides, including beams, plates, and shells, with the boundary conditions of the form (4.9) imposed on the free faces would be also of obvious interest. This would in fact seem to be a natural generalisation of the above mentioned example for the Rayleigh surface wave.

The general asymptotic scheme presented in Section 3 may also seemingly have potential applications outside the area of nonlocal elasticity. Firstly, we note applications for solids with localised near-surface inhomogeneities, such as functionally graded structures, see for example the review by Birman and Byrd (2007). There is also the possibility of adapting this scheme for long-wave dynamic analysis of vertically inhomogeneous foundations, see Muravskii (2001) and references therein.

Acknowledgements. Ph.D. studies of R. Chebakov were partly supported by Keele University. The support is gratefully acknowledged. The authors also highly appreciate valuable discussions with Dr D. A. Prikazchikov.

References

Abdollahi, R., Boroomand, B., 2013. Benchmarks in nonlocal elasticity defined by Eringen's integral model. International Journal of Solids and Structures 50 (18), 2758–2771.
URL http://www.sciencedirect.com/science/article/pii/S0020768313001844

Abdollahi, R., Boroomand, B., 2014. Nonlocal elasticity defined by Eringen's integral model: Introduction of a boundary layer method. International Journal of Solids and Structures 51 (9), 1758–1780.
URL http://www.sciencedirect.com/science/article/pii/S0020768314000201

Aghalovyan, L., 2014. Asymptotic Theory of Anisotropic Plates and Shells. World Scientific.

Arash, B., Wang, Q., 2012. A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes. Computational Materials Science 51 (1), 303–313.

Bazant, Z. P., Le, J.-L., Hoover, C. G., 2010. Nonlocal boundary layer (NBL) model: overcoming boundary condition problems in strength statistics and fracture analysis of quasibrittle materials. Fracture Mechanics of Concrete and Concrete StructuresŮRecent Advances in Fracture Mechanics of Concrete, B.-H. Oh, Ed., Korea Concrete Institute, Seoul, 135–143.

Benvenuti, E., Simone, A., 2013. One-dimensional nonlocal and gradient elasticity: closed-form solution and size effect. Mechanics Research Communications 48, 46–51.

Birman, V., Byrd, L. W., 2007. Modeling and analysis of functionally graded materials and structures. Applied Mechanics Reviews 60 (5), 195–216.

Craster, R., Kaplunov, J., Postnova, J., 2010. High-frequency asymptotics, homogenisation and localisation for lattices. The Quarterly Journal of Mechanics and Applied Mathematics 63 (4), 497–519.

Dai, H.-H., Kaplunov, J., Prikazchikov, D. A., 2010. A long-wave model for the surface elastic wave in a coated half-space. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences.

Dal Corso, F., Deseri, L., 2013. Residual stresses in random elastic composites: nonlocal micromechanics-based models and first estimates of the representative volume element size. Meccanica 48 (8), 1901–1923.

Del Piero, G., Owen, D. R., 2004. Multiscale Modeling in Continuum Mechanics and Structured Deformations. Vol. 447. Springer Science & Business Media.

Dell'Isola, F., Andreaus, U., Placidi, L., 2014. At the origins and in the vanguard of peridynamics, non-local and higher-gradient continuum mechanics: An underestimated and still topical contribution of gabrio piola. Mathematics and Mechanics of Solids, 1081286513509811.

Di Paola, M., Failla, G., Pirrotta, A., Sofi, A., Zingales, M., 2013. The mechanically based non-local elasticity: an overview of main results and future challenges. Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 371 (1993).

Di Paola, M., Failla, G., Zingales, M., 2009. Physically-based approach to the mechanics ofăstrongănon-local linear elasticity theory. Journal of Elasticity 97 (2), 103–130.
URL http://dx.doi.org/10.1007/s10659-009-9211-7

Di Paola, M., Zingales, M., 2008. Long-range cohesive interactions of non-local continuum faced by fractional calculus. International Journal of Solids and Structures 45 (21), 5642–5659.
URL http://www.sciencedirect.com/science/article/pii/S0020768308002461

Drugan, W. J., Willis, J. R., 1996. A micromechanics-based nonlocal constitutive equation and estimates of representative volume element size for elastic composites. Journal of the Mechanics and Physics of Solids 44 (4), 497–524.
URL http://www.sciencedirect.com/science/article/pii/0022509696000075

Eringen, A. C., 1966. A unified theory of thermomechanical materials. International Journal of Engineering Science 4 (2), 179–202.

Eringen, A. C., 1972. Linear theory of nonlocal elasticity and dispersion of plane waves. International Journal of Engineering Science 10, 425–435.

Eringen, A. C., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. Journal of Applied Physics 54 (9), 4703–4710.
URL http://scitation.aip.org/content/aip/journal/jap/54/9/10.1063/1.332803

Eringen, A. C., 1987. Theory of nonlocal elasticity and some applications. Res Mechanica 21 (4), 313–342.

Eringen, A. C., 2002. Nonlocal continuum field theories. Springer Science & Business Media.

Eringen, A. C., Edelen, D. G. B., 1972. On nonlocal elasticity. International Journal of Engineering Science 10 (3), 233–248.
URL http://www.sciencedirect.com/science/article/pii/0020722572900390

Eringen, A. C., Kim, B. S., 1977. Relation between non-local elasticity and lattice dynamics. Crystal Lattice Defects 7 (1), 51–57.

Goldenveizer, A. L., Kaplunov, J. D., Nolde, E. V., 1993. On Timoshenko-Reissner type theories of plates and shells. International Journal of Solids and Structures 30 (5), 675–694.
URL http://www.sciencedirect.com/science/article/pii/0020768393900297

Kroner, E., 1967. Elasticity theory of materials with long range cohesive forces. International Journal of Solids and Structures 3, 731–742.

Krumhansl, J., 1968. Some considerations of the relation between solid state physics and generalized continuum mechanics. In: Kroner, E. (Ed.), Mechanics of Generalized Continua. IUTAM Symposia. Springer Berlin Heidelberg, pp. 298–311.
URL http://dx.doi.org/10.1007/978-3-662-30257-6_37

Kunin, I., 1984. On foundations of the theory of elastic media with microstructure. International Journal of Engineering Science 22 (8 - 10), 969–978.
URL http://www.sciencedirect.com/science/article/pii/0020722584900983

Muravskii, G. B., 2001. Mechanics of Non-homogeneous and Anisotropic Foundations. Springer-Verlag Berlin Heidelberg.

Owen, D. R., Paroni, R., 2000. Second-order structured deformations. Archive for rational mechanics and analysis 155 (3), 215–235.

Owen, D. R., Paroni, R., 2015. Optimal flux densities for linear mappings and the multiscale geometry of structured deformations. Archive for Rational Mechanics and Analysis 218 (3), 1633–1652.
URL http://dx.doi.org/10.1007/s00205-015-0890-x

Panasenko, N., Bakhvalov, N. S., 1989. Homogenization: Averaging processes in periodic media: Mathematical problems in the mechanics of composite materials.

Peddieson, J., Buchanan, G. R., McNitt, R. P., 2003. Application of nonlocal continuum models to nanotechnology. International Journal of Engineering Science 41 (3), 305–312.

Polizzotto, C., 2001. Nonlocal elasticity and related variational principles. International Journal of Solids and Structures 38 (42), 7359–7380.

Rogula, D., 1982. Introduction to nonlocal theory of material media. In: Rogula, D. (Ed.), Nonlocal Theory of Material Media. Vol. 268 of International Centre for Mechanical Sciences. Springer

Vienna, pp. 123–222.
URL http://dx.doi.org/10.1007/978-3-7091-2890-9_3

Sanchez-Palencia, E., 1980. Non-homogeneous media and vibration theory. In: Non-homogeneous media and vibration theory. Vol. 127.

Schwartz, M., Niane, N., Njiwa, R. K., 2012. A simple solution method to 3D integral nonlocal elasticity: Isotropic-bem coupled with strong form local radial point interpolation. Engineering Analysis with Boundary Elements 36 (4), 606–612.

Zingales, M., 2011. Wave propagation in 1D elastic solids in presence of long-range central interactions. Journal of Sound and Vibration 330 (16), 3973–3989.
URL http://www.sciencedirect.com/science/article/pii/S0022460X10007133