LOCALIZED BULGING OF ROTATING ELASTIC CYLINDERS AND TUBES

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ABSTRACT. We investigate axially symmetric localized bulging of an incompressible hyperelastic circular solid cylinder or tube that is rotating about its axis of symmetry with angular velocity ω. For such a solid cylinder, the homogeneous primary deformation is completely determined by the axial stretch λ_z, and it is shown that the bifurcation condition is simply given by $d\omega/d\lambda_z = 0$ if the resultant axial force F is fixed. For a tube that is shrink-fitted to a rigid circular cylindrical spindle, the azimuthal stretch λ_a on the inner surface of the tube is specified and the deformation is again completely determined by the axial stretch λ_z although the deformation is now inhomogeneous. For this case it is shown that with F fixed the bifurcation condition is also given by $d\omega/d\lambda_z = 0$. When the spindle is absent (the case of unconstrained rotation), we also allow for the possibility that the tube is additionally subjected to an internal pressure P. It is shown that with P fixed, and ω and F both viewed as functions of λ_a and λ_z, the bifurcation condition for localized bulging is that the Jacobian of ω and F should vanish. Alternatively, the same bifurcation condition can be derived by fixing ω and setting the Jacobian of P and F to zero. Illustrative numerical results are presented using the Ogden and Gent material models.

1. INTRODUCTION

In a series of classic papers by Haughton and Ogden (1979a, b; 1980a, b, c) on the periodic buckling of hyperelastic circular solid cylinders or cylindrical tubes described by a general strain-energy function, two types of loading were considered separately. The first type consists of an internal pressure and an end thrust, and the second type consists of rotation about the axis of symmetry and an end thrust. The first situation has recently been reexamined with a view to characterizing axially symmetric localized bulging, motivated by the fact that in some loading regimes localized bulging is usually observed first but had previously not been fully understood. The second situation is now reexamined in the current paper, motivated by similar considerations.

Deformation of a rotating hyperelastic cylinder or tube is one of the first problems solved using the continuum mechanics theory (Green and Zerna 1968) with results used to test the validity of constitutive assumptions. For instance, when the neo-Hookean material model is used, the resulting axial stretch λ_z in a solid cylinder rotating with angular velocity ω is given by

$$\rho\omega^2 = \frac{4\mu}{A^2}(1 - \lambda_z^3), \tag{1-1}$$

Key words and phrases. Localized bulging, bifurcation, rotating tubes, nonlinear elasticity.

This work was supported by the National Natural Science Foundation of China (Grant Nos 11526141 and 11372212). Ali Althobaiti wishes to thank Taif University and the Saudi Arabian Ministry of Higher Education for their financial support during his PhD studies.

where μ, ρ, and A are the ground state shear modulus, material density and undeformed radius, respectively. The above result predicts that when $\rho\omega^2$ approaches the finite value $4\mu/A^2$, the stretch tends to zero, which is clearly un-physical, thus showing the inadequacy of the neo-Hookean material model in the large deformation regime. The question of the existence of a unique homogeneous (shape-preserving) solution valid for all values of ω was then examined by Chadwick *et al* (1977) using the newly proposed Ogden material model at that time. In a parallel study, the question of an axisymmetric bifurcation for a rotating cylinder was studied by Patterson and Hill (1977) using the neo-Hookean material model. They showed that such a bifurcation would take place before ω reaches the value corresponding to $\lambda = 0$. The bifurcation condition has an explicit expression and was derived in a similar manner to that used by Wilkes (1955) for studying the buckling of a cylinder or tube under end thrust. A significant generalization was subsequently made by Haughton and Ogden (1980a, b, c) who considered all the possible periodic buckling modes (prismatic, axisymmetric or asymmetric) for a rotating cylinder or tube without restricting the material model in their general formulation although numerical results were presented only for the Ogden material model. In the last three studies, it was observed that the rotation speed, when viewed as a function of a stretch measure, may reach a maximum beyond which the primary deformation no longer exists. This is similar to the so-called limiting point instability which was receiving a lot of attention around the same period; see, e.g., Kanner and Horgan (2007) and the references therein. However, in both situations the connection between the limiting point instability and localized bulging bifurcation was not fully understood at that time. It is shown in the present paper that under a certain loading condition the existence of such an angular speed maximum is closely associated with axisymmetric bulging localized in the axial direction, a phenomenon rather similar to localized bulging in circular cylindrical tubes that are inflated by an internal pressure.

We follow the same strategy as in Fu *et al* (2016). After formulating the problem and summarizing the expressions for the angular speed ω, the internal pressure P and the resultant axial force F associated with the primary shape-preserving deformation in the next section, we first derive the bifurcation condition in Section 3, and then conjecture and verify that the bifurcation condition in each case can be written simply in terms of the derivatives of ω, F and/or P with respect to the stretches as stated in the Abstract. In Section 4 the bifurcation condition is solved and numerical results are presented when both the Gent and Ogden material models are used. The paper is concluded in Section 5 with a summary of our main results and a discussion of whether localized bulging under rotation could be observed experimentally.

2. Primary deformation

We first write down the solution for the primary shape-preserving deformation associated with an incompressible circular cylindrical tube that is rotating about its axis of symmetry with angular velocity ω. The corresponding results for a solid cylinder will be obtained by specialization. The tube is assumed to have inner radius A and outer radius B in the undeformed configuration, and these dimensions take the values a and b in the deformed configuration. The outer surface of the tube is traction-free, but its inner surface may in general be subjected to a hydrostatic pressure P or shrink-fitted to a rigid circular cylindrical spindle with a radius larger than A. It is also assumed that each plane end-face of the tube is subject to a resultant axial force F (e.g. $F = 0$ in which case the end faces are traction-free). In terms of cylindrical polar coordinates, the primary shape-preserving deformation is given by

$$r = \sqrt{\lambda_z^{-1}(R^2 - A^2) + a^2}, \quad \theta = \Theta + \omega t, \quad z = \lambda_z Z, \tag{2-1}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical polar coordinates in the undeformed and deformed configurations, respectively, t denotes time, and λ_z is a constant. The associated principal stretches are

$$\lambda_1 = \frac{r}{R} \equiv \lambda, \quad \lambda_2 = \lambda_z, \quad \lambda_3 = \frac{dr}{dR} = 1/(\lambda\lambda_z). \tag{2-2}$$

We assume that the constitutive behavior of the tube is described by a strain-energy function $W(\lambda_1, \lambda_2, \lambda_3)$. By integrating the only equilibrium equation in the r-direction, it can be shown that various quantities can be expressed in terms of the reduced strain-energy function w, and its derivatives w_1 and w_2, defined by

$$w(\lambda, \lambda_z) = W(\lambda, \lambda_z, \lambda^{-1}\lambda_z^{-1}), \quad w_1 = \partial w/\partial\lambda, \quad w_2 = \partial w/\partial\lambda_z; \tag{2-3}$$

see Haughton and Ogden (1980c). For instance, the normal stress on the inner surface is given by

$$\sigma_{33}(a) = \frac{1}{2}\rho\omega^2\lambda_z^{-1}(B^2 - A^2) - \int_{\lambda_b}^{\lambda_a} \frac{w_1}{\lambda^2\lambda_z - 1} d\lambda, \tag{2-4}$$

where ρ is the material density, the two limits λ_a and λ_b are defined by

$$\lambda_a = \frac{a}{A}, \quad \lambda_b = \frac{b}{B},$$

and are related to each other, through incompressibility, by

$$\lambda_a^2\lambda_z - 1 = \frac{B^2}{A^2}(\lambda_b^2\lambda_z - 1). \tag{2-5}$$

We observe that with λ_b eliminated using the above relation, the deformation is completely determined by the two stretches λ_a and λ_z.

The expression (2-4) suggests the introduction of a non-dimensional quantity Γ defined by

$$\Gamma = \rho\omega^2 B^2/\mu, \tag{2-6}$$

where μ denotes the ground-state shear modulus. In the rest of this paper, we shall assume that the strain-energy function, stress components and pressure have all been scaled by μ, forces by μB^2, and the radii A, a, b by B. As a result of non-dimensionalization, equation (2-4) is now replaced by

$$\sigma_{33}(a) = \frac{1}{2}\Gamma\lambda_z^{-1}(1 - A^2) - \int_{\lambda_b}^{\lambda_a} \frac{w_1}{\lambda^2\lambda_z - 1} d\lambda, \tag{2-7}$$

and the resultant of σ_{22} at any cross section is given by

$$\tilde{F}_0(\lambda_a, \lambda_z, \Gamma) \equiv 2\pi \int_a^b \sigma_{22} r\,dr = \frac{1}{4}\pi\Gamma\lambda_z^{-2}(1 - A^2)^2 + \pi a^2 \int_{\lambda_b}^{\lambda_a} \frac{w_1}{\lambda^2\lambda_z - 1} d\lambda$$

$$+\pi A^2(\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} \frac{2\lambda_z w_2 - \lambda w_1}{(\lambda^2\lambda_z - 1)^2}\lambda d\lambda, \tag{2-8}$$

where the first equation serves to define the function $\tilde{F}_0$. Finally, the normal stress component σ_{33} in the r-direction is given by

$$\sigma_{33}(r) = \frac{1}{2}\Gamma(b^2 - r^2) - \int_{\lambda_b}^{\lambda} \frac{w_1}{\lambda^2\lambda_z - 1}d\lambda. \tag{2-9}$$

This expression recovers (2-7) on evaluation at $r = a$ followed by the use of the identity $(b^2 - a^2)\lambda_z = B^2 - A^2$. We can now specialize the above formulae to three different loading and geometry conditions.

Firstly, consider a tube under the combined action of an internal pressure, rotation and an end thrust (referred to hereafter as the unconstrained case). Denoting the internal pressure by P, we have $\sigma_{33}(a) = -P$, and (2-7) then becomes

$$P = \tilde{P}(\lambda_a, \lambda_z) \equiv -\frac{1}{2}\Gamma\lambda_z^{-1}(1 - A^2) + \int_{\lambda_b}^{\lambda_a} \frac{w_1}{\lambda^2\lambda_z - 1}d\lambda, \tag{2-10}$$

or equivalently,

$$\Gamma = \tilde{\Gamma}(\lambda_a, \lambda_z) \equiv \frac{2\lambda_z}{1 - A^2}\int_{\lambda_b}^{\lambda_a} \frac{w_1}{\lambda^2\lambda_z - 1}d\lambda - \frac{2\lambda_z P}{1 - A^2}, \tag{2-11}$$

where the second expression in each equation defines the functions $\tilde{P}$ and $\tilde{\Gamma}$, respectively. To simplify notation, we have not shown explicitly the dependence of $\tilde{P}$ on Γ or $\tilde{\Gamma}$ on P.

The resultant axial force at any cross section, that is to be balanced by the net force F applied at each plane end-face, is given by

$$F = \tilde{F}(\lambda_a, \lambda_z) \equiv 2\pi\int_a^b \sigma_{22} r dr - \pi a^2 P$$

$$= \frac{1}{4}\pi\Gamma\lambda_z^{-1}(a^2 + b^2)(1 - A^2) + \pi A^2(\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} \frac{2\lambda_z w_2 - \lambda w_1}{(\lambda^2\lambda_z - 1)^2}\lambda d\lambda, \tag{2-12}$$

where the second equation defines the function $\tilde{F}$ which should be compared with the $\tilde{F}_0$ defined by (2-8). We note that the Γ in the expression for $\tilde{F}$ could be eliminated in favour of P with the use of (2-11). Thus, whenever $\tilde{F}$ is partially differentiated, it can be either Γ or P that is fixed. This will always be indicated explicitly.

Once the geometry of the tube is specified, either P or Γ can be assumed to take a dominant role together with F. If, for instance, P is specified and is assumed to take a passive role (by being sufficiently small), then Γ and F can be viewed as functions of the two stretches λ_a and λ_z, and we expect the following Jacobian to play a role in the characterization of localized bulging:

$$\Omega_u(\lambda_a, \lambda_z) \equiv J(\tilde{\Gamma}, \tilde{F}) = \frac{\partial\tilde{\Gamma}}{\partial\lambda_a}\frac{\partial\tilde{F}}{\partial\lambda_z} - \frac{\partial\tilde{\Gamma}}{\partial\lambda_z}\frac{\partial\tilde{F}}{\partial\lambda_a}, \tag{2-13}$$

where the first equation defines the function $\Omega_u(\lambda_a, \lambda_z)$ with the subscript u signifying "unconstrained". Similarly, if Γ is assumed to take a passive role, then the Jacobian $J(\tilde{P}, \tilde{F})$ can be defined. However, it can be shown that this Jacobian is a non-zero multiple of $J(\tilde{\Gamma}, \tilde{F})$ when the connection (2-10) is used.

The case previously studied by Fu *et al* (2016) can now be viewed a special case, corresponding to $\Gamma \equiv 0$, of the current more general formulation. The observations made in the latter paper about $J(\tilde{P}, \tilde{F})$ can be extended to the case when Γ is non-zero but is held fixed. In particular, it can be shown that when F and Γ are both fixed, the pressure will reach a maximum precisely when $J(\tilde{P}, \tilde{F}) = 0$. In a similar manner, we note that if both P and F are held fixed when Γ is increased gradually, $\tilde{F}(\lambda_a, \lambda_z) = F$ would define λ_z as a function of λ_a, and we then have

$$\frac{\partial \tilde{F}}{\partial \lambda_a} + \frac{\partial \tilde{F}}{\partial \lambda_z} \frac{d\lambda_z}{d\lambda_a} = 0,$$

and

$$\frac{d\tilde{\Gamma}}{d\lambda_a} = \frac{\partial \tilde{\Gamma}}{\partial \lambda_a} + \frac{\partial \tilde{\Gamma}}{\partial \lambda_z} \frac{d\lambda_z}{d\lambda_a} = (\frac{\partial \tilde{F}}{\partial \lambda_z})^{-1} J(\tilde{\Gamma}, \tilde{F}).$$

Thus, Γ reaches a maximum when the Jacobian $J(\tilde{\Gamma}, \tilde{F})$ vanishes. We emphasize that this correspondence is lost when, for instance, it is the λ_z that is held fixed in rotating the tube. Drawing upon the results of Fu *et al* (2016), we may then further conjecture that when the inner surface is traction-free or subjected to a hydrostatic pressure P the bifurcation condition for localized bulging is simply $J(\tilde{\Gamma}, \tilde{F}) = 0$, whether it is the F or λ_z that is fixed in rotating the tube. We shall verify in the next section that this is indeed the case.

Next, consider the case when a tube is mounted on a rigid circular cylindrical spindle with a radius larger than A, which is the shrink-fit case discussed in Chadwick *et al* (1977). We assume that the contact is smooth so that at the inner surface the stretch λ_a is specified and the shear stress components are negligible. When such a tube is rotated, the expression (2-4) can be used to compute the contact pressure, and the resultant $\tilde{F}_0(\lambda_a, \lambda_z, \Gamma)$ of σ_{22} given by (2-8) is now a function of the only variable stretch λ_z. It will be shown in the next section that the bifurcation condition for localized bulging is simply

$$\Omega_c(\lambda_a, \lambda_z, \Gamma) \equiv \frac{\partial \tilde{F}_0}{\partial \lambda_z} = 0, \tag{2-14}$$

where the first equation defines the function $\Omega_c(\lambda_a, \lambda_z, \Gamma)$ with the subscript c signifying "constrained". Suppose that the equation $\tilde{F}_0(\lambda_a, \lambda_z, \Gamma) = F$, with $\tilde{F}_0$ given by (2-8), is solved for Γ and the result is denoted by $\Gamma = \tilde{\Gamma}_0(\lambda_a, \lambda_z, F)$. Then it can also be shown that the above bifurcation condition is equivalent to $\partial \tilde{\Gamma}_0 / \partial \lambda_z = 0$.

Finally, in the special case of a solid cylinder (A=a=0), the three principal stretches given by (2-2) reduce to

$$\lambda_1 = \lambda_z^{-1/2}, \quad \lambda_2 = \lambda_z, \quad \lambda_3 = \lambda_z^{-1/2}, \tag{2-15}$$

which are all independent of r. Then in terms of the reduced strain-energy function $\hat{w}$ defined by

$$\hat{w}(\lambda_z) = W(\lambda_z^{-1/2}, \lambda_z, \lambda_z^{-1/2}), \tag{2-16}$$

the principal stretch λ_z is determined by

$$\Gamma = 4\lambda_z^2 (\frac{F}{\pi} - \hat{w}'), \tag{2-17}$$

where the prime denotes differentiation with respect to the argument λ_z. If F is fixed, then setting $d\Gamma/\lambda_z = 0$ would yield, after F has been eliminated with the use of (2-17), the condition

$$\Omega_s(\lambda_z) \equiv 2\lambda_z^3 \hat{w}'' - \Gamma = 0, \tag{2-18}$$

where the first relation defines the function $\Omega_s(\lambda_z)$ with the subscript s signifying "solid". On the other hand, if Γ is fixed instead, then setting $dF/\lambda_z = 0$ would again yield the same condition (2-18). It will be shown in the next section that this is in fact the bifurcation condition for localized bulging.

3. Bifurcation conditions for localized bulging

To investigate the axisymmetric localized bulging of the finitely deformed configurations determined in the previous section, we consider an incremental displacement field $\dot{\boldsymbol{x}}$ given by

$$\dot{\boldsymbol{x}} = u(r,z)\boldsymbol{e}_r + v(r,z)\boldsymbol{e}_z, \tag{3-1}$$

where $u(r,z)$ and $v(r,z)$ are the incremental displacements in the r- and z-directions, respectively. The incremental equation of motion takes the form

$$\text{div}\, \boldsymbol{\chi}^T = -\Gamma u \boldsymbol{e}_r, \tag{3-2}$$

where the incremental stress tensor $\boldsymbol{\chi}$ is defined by the following components relative to the orthonormal basis $\{\boldsymbol{e}_\theta, \boldsymbol{e}_z, \boldsymbol{e}_r\}$:

$$\chi_{ij} = \mathcal{B}_{jilk}\eta_{kl} + \bar{p}\eta_{ji} - p^*\delta_{ji}. \tag{3-3}$$

In the above expression, the $\mathcal{B}_{jilk}$'s are the incremental elastic moduli whose expression in terms of the principal stretches can be found in Haughton and Ogden (1979a), $\bar{p}$ and p^* are, respectively, the primary and incremental pressures associated with the constraint of incompressibility, and the tensor $\boldsymbol{\eta}$ is given by

$$\boldsymbol{\eta} = \begin{bmatrix} \frac{u}{r} & 0 & 0 \\ 0 & v_z & v_r \\ 0 & u_z & u_r \end{bmatrix}, \quad \text{with } v_z \equiv \frac{\partial v}{\partial z}, \quad v_r \equiv \frac{\partial v}{\partial r} \text{ etc.} \tag{3-4}$$

The equation of motion is to be supplemented by the incompressibility condition

$$\text{tr}\, \boldsymbol{\eta} = u_r + v_z + \frac{u}{r} = 0, \tag{3-5}$$

and is to be solved subject to appropriate boundary conditions on $r = a, b$. On the outer boundary $r = b$, we impose the traction-free boundary condition

$$\boldsymbol{\chi}\boldsymbol{n} = \boldsymbol{0} \quad \text{on } r = b \tag{3-6}$$

for all the three cases under consideration, where $\boldsymbol{n}$ denotes the unit normal to the surface. For a solid cylinder, this boundary condition is supplemented by the condition that the solution must be bounded at $r = 0$. For the case when a tube is subjected to an internal (hydrostatic) pressure P, the boundary condition on $r = a$ is given by

$$\boldsymbol{\chi}\boldsymbol{n} = P\boldsymbol{\eta}^T\boldsymbol{n} \quad \text{on } r = a. \tag{3-7}$$

Finally, for the shrink-fit case, the boundary condition on $r = a$ is given by

$$\chi_{23} = 0, \quad u = 0 \quad \text{on} \;\; r = a. \tag{3-8}$$

As explained in Fu *et al* (2016), the bifurcation condition can be derived by first looking for a solution of the form

$$u = f(r)\mathrm{e}^{\alpha z}, \quad v = g(r)\mathrm{e}^{\alpha z}, \quad p^* = h(r)\mathrm{e}^{\alpha z}. \tag{3-9}$$

On substituting these expressions into the incremental equilibrium equations and then eliminating $g(r)$ and $h(r)$ in favor of $f(r)$, we find that $f(r)$ satisfies the single fourth-order ordinary differential equation

$$\begin{aligned} r^4\left\{r^{-1}[r^{-1}\mathcal{B}_{3232}(r^2 f'' + rf' - f)]'\right\}' &+ \alpha^2 r^2\left\{r[r(\mathcal{B}_{2222} + \mathcal{B}_{3333} - 2\mathcal{B}_{2233} - 2\mathcal{B}_{3223})f']'\right. \\ &+(r^2\sigma_{33}'' - r^2\mathcal{B}_{3232}'' + r\mathcal{B}_{2222}' + r\mathcal{B}_{1133}' - r\mathcal{B}_{1122}' - r\mathcal{B}_{2233}' - r\mathcal{B}_{3223}' \\ &\left.+2\mathcal{B}_{1122} + 2\mathcal{B}_{3223} - \mathcal{B}_{1111} - \mathcal{B}_{2222} + r^2\Gamma)f\right\} + \alpha^4 r^4\mathcal{B}_{2323}f = 0. \end{aligned} \tag{3-10}$$

This corresponds to Haughton and Ogden (1980c)'s equation (49) with α replaced by iα. In a similar manner, the two boundary conditions (3-6) and (3-7) yield

$$r^2 f'' + rf' - (\alpha^2 r^2 + 1)f = 0 \quad \text{on} \;\; r = a, b, \tag{3-11}$$

and

$$\begin{aligned} r^2[r^{-1}\mathcal{B}_{3232}(r^2 f'' + rf' - f)]' &+ \alpha^2 r^3(\mathcal{B}_{2222} + \mathcal{B}_{3333} - 2\mathcal{B}_{2233} - 2\mathcal{B}_{3223} + \mathcal{B}_{3232})f' \\ -\alpha^2 r^2(r\sigma_{33}' - r\mathcal{B}_{3232}' &+ \mathcal{B}_{2222} + \mathcal{B}_{1133} - \mathcal{B}_{1122} - \mathcal{B}_{2233} - \mathcal{B}_{3223})f = 0 \quad \text{on} \;\; r = a, b. \end{aligned} \tag{3-12}$$

The last boundary condition (3-12) on $r = a$ corresponds to (3-7) applied in the normal direction. For the shrink-fitting case, this is replaced by

$$f(a) = 0, \tag{3-13}$$

and as a result the boundary condition (3-11), which corresponds to $\chi_{23} = 0$, reduces to

$$af''(a) + f'(a) = 0. \tag{3-14}$$

Because of the translational invariance in terms of z, $\alpha = 0$ is always an eigenvalue of the above eigenvalue problem. For sufficiently small values of Γ, P and F, none of the other eigenvalues can be pure imaginary since such eigenvalues give rise to bifurcation modes that are sinusoidal in the z-direction, and we only expect such modes to appear for sufficiently large values of Γ, P and F. We note that Haughton and Ogden (1980a, b, c)'s analysis is concerned with the conditions under which such bifurcation modes would appear. Since our current analysis is concerned with a bifurcation mode that is localized in the axial direction, as a first attempt we may assume that the tube or solid cylinder is infinitely long so that boundary conditions at the two end-faces need not to be considered. This is essentially the "perfect" bifurcation case (i.e. bifurcation in the absence of any imperfections). Effects of finite length as well as material inhomogeneity and/or non-uniform wall thickness can all be considered as imperfections. Since localized bulging is in general a sub-critical bifurcation, it is expected that in the presence of imperfections, the critical value of angular speed will be significantly lower than the value determined in the current paper. For an illustration of the effect of imperfections, we refer to Fu and Xie (2012).

For the problem under consideration, there exist two *real* eigenvalues of the form $\alpha = \pm\alpha_1$ that are closest to $\alpha = 0$ when Γ, P and F are sufficiently small. These two eigenvalues would move along the real axis towards the origin as Γ and/or P is increased gradually. According to the dynamical systems theory, a necessary condition for a localized bulging bifurcation to take place is when α_1 vanishes, making zero a triple eigenvalue; see, e.g., Kirchgässner (1982) or Haragus and Iooss (2011). This necessary condition is now derived for the three different cases defined earlier.

3A. **Solid cylinder.** For the case of a solid cylinder, all the elastic moduli are constants and the reduced eigenvalue problem can be solved analytically. Adapting Haughton and Ogden (1980a) results slightly, we obtain the following equation satisfied by all the eigenvalues of α:

$$K_1 - K_2 = 0, \tag{3-15}$$

where K_1 and K_2 are defined by

$$\nu_\beta(\nu_\beta^2+1)I_1(\nu_\beta\alpha b)K_\beta = \alpha b(\nu_\beta^2\mathcal{B}_{1212}+\mathcal{B}_{2121})I_0(\nu_\beta\alpha b) - \nu_\beta(2\mathcal{B}_{1313}+\Gamma b^2)I_1(\nu_\beta\alpha b), \tag{3-16}$$

with ν_1, ν_2 denoting the two positive roots of the bi-quadratic equation

$$\nu^4\mathcal{B}_{1212} - \nu^2(\mathcal{B}_{1111}+\mathcal{B}_{2222}-2\mathcal{B}_{1122}-2\mathcal{B}_{1221}) + \mathcal{B}_{2121} = 0. \tag{3-17}$$

The condition for zero to become a triple eigenvalue can be obtained by expanding (3-15) in terms of α and then setting the leading term to zero. The result is

$$(2\mathcal{B}_{1313} - 2\mathcal{B}_{1212} + \Gamma b^2)\nu_1^2\nu_2^2 - 2\mathcal{B}_{2121}(\nu_1^2+\nu_2^2) - 2\mathcal{B}_{2121} = 0,$$

which, on using (3-17), can be reduced to

$$\Gamma b^2 - 2\mathcal{B}_{1111} + 4\mathcal{B}_{1122} - 4\mathcal{B}_{1212} + 4\mathcal{B}_{1221} + 2\mathcal{B}_{1313} - 2\mathcal{B}_{2222} = 0. \tag{3-18}$$

With the use of the expressions for the elastic moduli given by Haughton and Ogden (1979a), we have verified that (3-18) can be reduced to (2-18). Thus, we conclude that for a rotating solid cylinder, a localized bulge will initiate when the rotation speed ω or the axial force given by (2-17) reaches a maximum.

3B. **Unconstrained tube.** We next consider the case of an unconstrained tube that is subjected to the combined action of rotation, internal inflation and an axial force. An inspection of the associated eigenvalue problem governed by (3-10), (3-11) and (3-12) shows that it can be obtained from the case with $\Gamma = 0$ by the simple substitution

$$\mathcal{B}_{1111} \to \mathcal{B}_{1111} - \Gamma r^2. \tag{3-19}$$

As a result, the exact bifurcation condition can be derived following the same procedure as in Fu *et al* (2016). Guided by the results in the latter paper, we may further conjecture that with the above substitution, the new bifurcation condition should be equivalent to $\Omega_u(\lambda_a, \lambda_z) = 0$, where $\Omega_u(\lambda_a, \lambda_z)$ is defined by (2-13). We have checked to verify that this is indeed the case. Furthermore, in the thin-wall limit, with the aid of an expansion procedure adapted from Fu *et al* (2016), we find that the exact bifurcation condition to leading order reduces to

$$\lambda_m(w_1 - \lambda_z w_{12})^2 + \lambda_z^2 w_{22}(w_1 - \lambda_1 w_{11}) + \lambda_m^3\Gamma^2 + 2\lambda_m^2(\lambda_z w_{12} - w_1)\Gamma = 0, \tag{3-20}$$

where λ_m denotes the azimuthal stretch in the mid-surface, and all the partial derivatives are now evaluated at $\lambda = \lambda_m$. The associated leading-order expressions for Γ and F take the form

$$\Gamma = w_1/\lambda_m - \lambda_z P_0, \quad F = \pi(2w_2 - P_0\lambda_m^2), \tag{3-21}$$

where $P_0 = P/\epsilon$, ϵ being the wall thickness scaled by the averaged radius. If $(3\text{-}21)_1$ is used to eliminate Γ in (3-20), we obtain the alternative bifurcation condition

$$\lambda_m(w_{11}w_{22} - w_{12}^2) - w_1 w_{22} + P_0\lambda_m^2(2w_{12} - \lambda_m P_0) = 0, \tag{3-22}$$

which is valid if P_0 is held fixed in rotating the tube. As a useful check, this leading-order bifurcation condition can also be obtained from $J(\Gamma, F) = 0$ when the leading-order expressions (3-21) are used.

As expected, when $\Gamma = 0$, (3-20) reduces to its counterpart for the pure inflation case given in Fu *et al* (2008), and it is known that this reduced bifurcation condition has a solution that defines λ_z as a function of λ_m for most of the commonly used strain-energy functions. In contrast, if we set $P_0 = 0$ in (3-22), the existence of solution of the reduced bifurcation condition depends very much on the material model used: it again has a solution when the Ogden material model is used, but it does not have a solution when the Gent material model is used. This difference carries over even when finite wall thickness is considered, which will be discussed further in the next section. We also observe that equation (3-22) with $P_0 = 0$ is the same as Haughton and Ogden's (1980c) equation (63) which emerged as the limit of the condition of bifurcation into axially symmetric periodic modes in an infinitely long tube. This equation reappeared as equation (71) in the same paper where it was observed as characterizing the turning point of ω when the axial force F is held fixed.

3C. **Constrained tube.** Finally, we consider the shrink-fit case for which the applicable boundary conditions at $r = a, b$ are (3-13) and (3-14), and (3-11) and (3-12), respectively. Again, the bifurcation condition is the condition under which zero becomes a triple eigenvalue. To derive this condition, we expand the stretches as

$$\lambda_a = \lambda_a^{(0)} + \alpha^2\lambda_a^{(1)} + \cdots, \quad \lambda_z = \lambda_z^{(0)} + \alpha^2\lambda_z^{(1)} + \cdots, \tag{3-23}$$

and look for a regular perturbation solution of the form

$$f(r) = f^{(0)}(r) + \alpha^2 f^{(1)}(r) + \cdots, \tag{3-24}$$

where the constants $\lambda_a^{(i)}, \lambda_z^{(i)}$ $(i = 0, 1, \cdots)$ and functions $f^{(i)}(r)$ $(i = 0, 1, \cdots)$ are to be determined. We are basically assuming that there is a small real eigenvalue α and then determining the required values of the stretches that support such a small eigenvalue. If such a non-trivial solution can be found, then the leading order values $\lambda_a^{(0)}$ and $\lambda_z^{(0)}$ are the stretch values at which zero becomes a triple eigenvalue.

On substituting (3-24) into (3-10) and equating the coefficients of α^0 and α^2, we obtain a homogeneous equation for $f^{(0)}(r)$ and an inhomogeneous equation for $f^{(1)}(r)$. These equations are the same as their counterparts derived in Fu *et al* (2016) except for the substitution (3-19).

We thus have

$$f^{(0)}(r) = c_1 r + c_2 \frac{1}{r} + c_3 \kappa_1(r) + c_4 \kappa_2(r), \tag{3-25}$$

and

$$f^{(1)}(r) = d_1 r + d_2 \frac{1}{r} + d_3 \kappa_1(r) + d_4 \kappa_2(r) + c_1 \kappa_3(r) + c_2 \kappa_4(r) + c_3 \kappa_5(r) + c_4 \kappa_6(r), \tag{3-26}$$

where c_i and d_i (i = 1, 2, 3, 4) are constants and the expressions for $\kappa_i(r)$ (i=1, …, 6) are not written out here for the sake of brevity. We observe that the last four terms in (3-26) are simply particular integrals. The boundary conditions can similarly be expanded. On substituting (3-24) together with (3-25) and (3-26) into the leading- and second-order boundary conditions, we find that $c_3 = c_4 = 0$, $c_2 = -ac_1$, $d_2 = -ad_1$, and that the three constants c_1, d_3, d_4 satisfy three homogeneous linear equations. For a non-trivial solution, we set the determinant of its coefficient matrix to zero, thus obtaining the condition

$$\Omega(\lambda_a^{(0)}, \lambda_z^{(0)}) = 0, \tag{3-27}$$

where

$$\Omega(\lambda_a, \lambda_z) = (a^2b^2 - b^4)F_1 + (a^2b^4 - a^4b^2)F_2 + 2b^2F_3 - 2a^2b^2F_4 + (a^4 - b^4)\sigma_3(b)$$

$$+b^4(\mathcal{B}_{1122}(b) - \mathcal{B}_{1133}(b) - 2\mathcal{B}_{2222}(b) + 3\mathcal{B}_{2233}(b) + 2\mathcal{B}_{3223}(b) - 2\mathcal{B}_{3232}(b) - \mathcal{B}_{3333}(b))$$

$$+a^2b^2(2\mathcal{B}_{1133}(b) + 2\mathcal{B}_{2222}(b) + 2\mathcal{B}_{3232}(b) - 2\mathcal{B}_{1122}(b) - 2\mathcal{B}_{2233}(b) - 2\mathcal{B}_{3223}(b))$$

$$+b(a^2 - b^2)^2(\mathcal{B}'_{3223}(b) + \bar{p}'(b)) + a^4(\mathcal{B}_{1122}(b) - \mathcal{B}_{1133}(b) - \mathcal{B}_{2233}(b) + \mathcal{B}_{3333}(b)).$$

In the last expression, the constants $F_1, \ldots, F_4$ are given by

$$F_1 = \int_a^b \omega_1(t)dt, \quad F_3 = \int_a^b t\left(\int_a^t \omega_1(s)ds\right)dt,$$

$$F_2 = \int_a^b \omega_2(t)dt, \quad F_4 = \int_a^b t\left(\int_a^t \omega_2(s)ds\right)dt.$$

$$\omega_1(r) = \mathcal{B}'_{1122} - \mathcal{B}'_{1133} + 3\mathcal{B}'_{2233} - 2\mathcal{B}'_{2222} - \mathcal{B}'_{3333} + 3\mathcal{B}'_{3223} + r(\mathcal{B}''_{3223} + \bar{p}'')$$

$$+\frac{1}{r}(\mathcal{B}_{1111} - \Gamma r^2 - 2\mathcal{B}_{1122} + 2\mathcal{B}_{2233} - \mathcal{B}_{3333}),$$

$$\omega_2(r) = \frac{1}{r}(\mathcal{B}''_{3223} + \bar{p}'') + \frac{1}{r^2}(\mathcal{B}'_{1122} - \mathcal{B}'_{1133} - \mathcal{B}'_{2233} - \mathcal{B}'_{3333} - \mathcal{B}'_{3223})$$

$$+\frac{1}{r^3}(\mathcal{B}_{1111} - \Gamma r^2 - 2\mathcal{B}_{1122} + 2\mathcal{B}_{2233} - \mathcal{B}_{3333}).$$

We may conjecture that the bifurcation condition (3-27) is equivalent to (2-14). This is verified numerically to be indeed the case.

4. Numerical results

We present some representative numerical results by considering two commonly used material models for rubber-like materials, the Ogden and Gent material models; see Ogden (1972) and Gent (1996). The associated strain-energy function is given by

$$W = \mu \sum_{r=1}^{3} \mu_r^* (\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3)/\alpha_r, \tag{4-1}$$

and

$$W = -\frac{\mu}{2} J_m \ln(1 - \frac{J_1}{J_m}), \quad J_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3, \tag{4-2}$$

respectively, where μ is the ground state shear modulus,

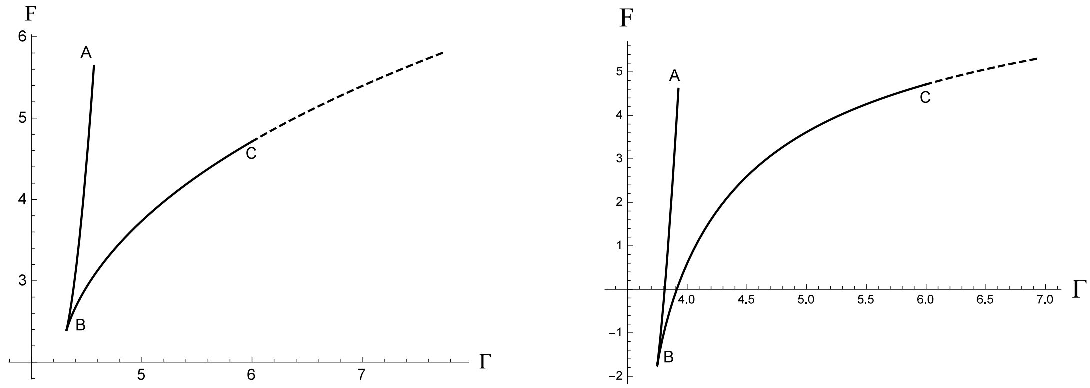
$$\alpha_1 = 1.3, \quad \alpha_2 = 5.0, \quad \alpha_3 = -2.0, \quad \mu_1^* = 1.491, \quad \mu_2^* = 0.003, \quad \mu_3^* = -0.023,$$

and J_m is a material constant which we shall take to be 97.2 following Gent (1996). All our numerical computations were carried out using *Mathematica* version 10.2.0.0 (1991).

4A. **Solid cylinder.** We assume that a net axial force F is specified and we wish to determine the value of Γ at which localized bulging may take place. In Figure 1, we have shown the relation between Γ and F when the bifurcation condition (2-18) is satisfied. The relation is obtained by varying λ_z continuously from 0.2 to 1.2 and computing the associated Γ and F using (2-17) and (3-15). It is seen that both curves contain a cusp point, corresponding to the fact that the Γ given by $\Gamma = 2\lambda_z^3 \hat{w}''$, defined by the bifurcation condition (2-18), attains a minimum at $\lambda_z = 0.39$ for the Gent material, and $\lambda_z = 0.20$ for the Ogden material; see Figure 2(a). In Figure 2(b) is shown the dependence of F on λ_z when localized bulging occurs. It is seen that there is a major difference between the predictions between the two material models: whereas the Ogden model predicts that localized bulging may take place when $F = 0$ and the associated critical value of $\Gamma = 3.81$ is attained at $\lambda_z = 0.15$, the Gent material model predicts that localized bulging can never take place when $F = 0$.

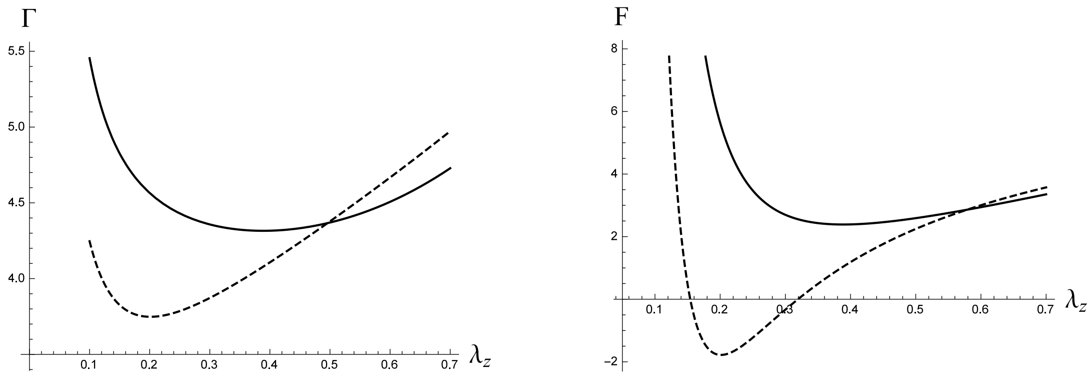
4B. **Unconstrained tube.** We first consider the simplest case when $P = 0$, and we determine the critical value of Γ at which localized bulging may occur. We have in mind two possible types of end conditions: either F or λ_z is fixed when Γ is increased. We note, however, that the solution of the bifurcation condition $\Omega_u(\lambda_a, \lambda_z) = 0$ is independent of the end conditions. It is found that when the Gent material model is used, this bifurcation condition does not have a solution, and so localized bulging can never occur however large the rotation speed is. This fact was already noted in the previous section in the thin-wall limit. In contrast, when the Ogden material model is used, $\Omega_u(\lambda_a, \lambda_z) = 0$ has a solution giving λ_z as a function of λ_a; see the solid line in Figure 3(a) where we have also shown a typical loading curve defined by $\tilde{F}(\lambda_a, \lambda_z) = 0$. As Γ is increased gradually from zero, loading starts from the point where $(\lambda_a, \lambda_z) = (1, 1)$ and traces down the loading curve $\tilde{F}(\lambda_a, \lambda_z) = 0$. Localized bulging would occur when this curve intersects the bifurcation curve $\Omega_u(\lambda_a, \lambda_z) = 0$ at

$$(\lambda_a, \lambda_z) = (2.313, 0.683),$$



(a) Gent material (b) Ogden material

FIGURE 1. Relation between Γ and F when localized bulging takes place. The solid and dashed parts on each curve corresponds to $\lambda_z < 1$ and $\lambda_z > 1$, respectively.



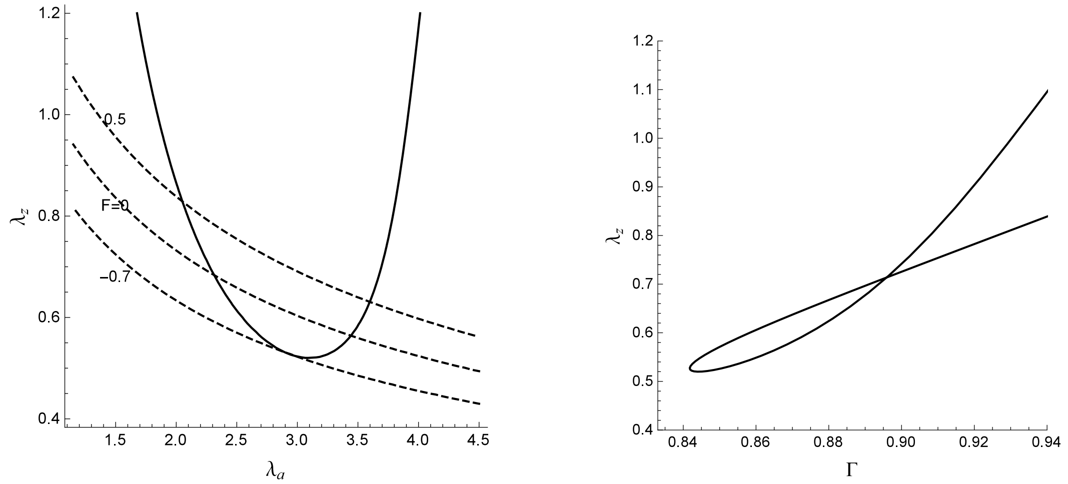
(a) (b)

FIGURE 2. Dependence of Γ and F on λ_z when localized bulging takes place. Solid and dashed lines correspond to the Gent and Ogden material models, respectively.

with the associated value of Γ equal to 0.885.

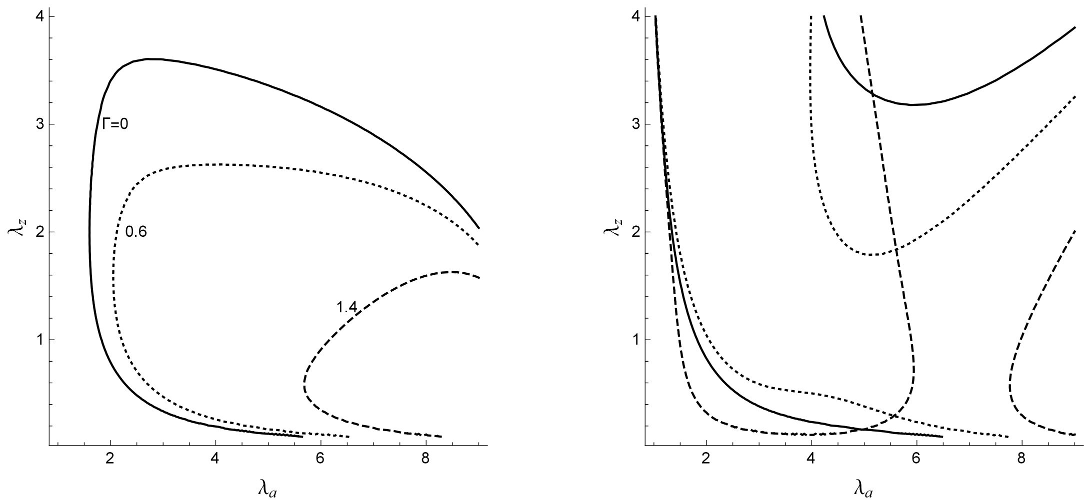
As a comparison, we have also shown in Figure 3(a) the loading paths associated with $F = 0.5$ and -0.7, respectively. It is seen that as F is increased from zero, the loading path is shifted upwards, whereas as F is decreased from zero, the loading paths is shifted downwards, eventually losing the intersections with the bifurcation curve when F is approximately equal to -0.7. This means that when $P = 0$, localized bulging is always possible when a stretching force is applied axially in addition to the rotation, but becomes impossible when a compressive force exceeding 0.7 in magnitude is applied.

To offer a different perspective on the bifurcation, we have shown in Figure 3(b) the critical value of λ_z as a function of Γ. This curve is obtained as follows. For each value of λ_z, the bifurcation condition $\Omega_u(\lambda_a, \lambda_z) = 0$ is solved to find the corresponding values of λ_a (there are



(a) (b)

FIGURE 3. Solution of the bifurcation condition $\Omega_u(\lambda_a, \lambda_z) = 0$, when the Ogden material is used, for a rotating tube with $A = 0.8$ and $P = 0$ in terms of (λ_a, λ_z) (left figure) and (Γ, λ_z) (right figure), respectively. The dashed lines represent the loading paths with $F = 0, 0.5, -0.7$, respectively.



(a) Gent material (b) Ogden material

FIGURE 4. Solution of the bifurcation condition $J(\tilde{P}, \tilde{F}) = 0$ for the different values of Γ indicated. The solid, dotted and dashed lines correspond to $\Gamma = 0, 0.2$ and 0.5, respectively.

zero, one or two such values, as can be seen from Figure 3(a)), and hence the values of Γ. This alterative plot of the bifurcation condition is particularly useful when it is the axial stretch that is fixed as the rotation speed is increased gradually. For each such axial stretch, λ_{z0} say, the corresponding critical value of Γ can simply be obtained from the leftmost intersection of the horizontal line $\lambda_z = \lambda_{z0}$ in this plot with the bifurcation curve. For instance, when λ_z is fixed at unity so that the tube is not allowed to contract axially as the rotation speed is increased, the associated critical value of Γ is equal to 0.997. It is seen that the critical value of Γ is an increasing function of λ_z.

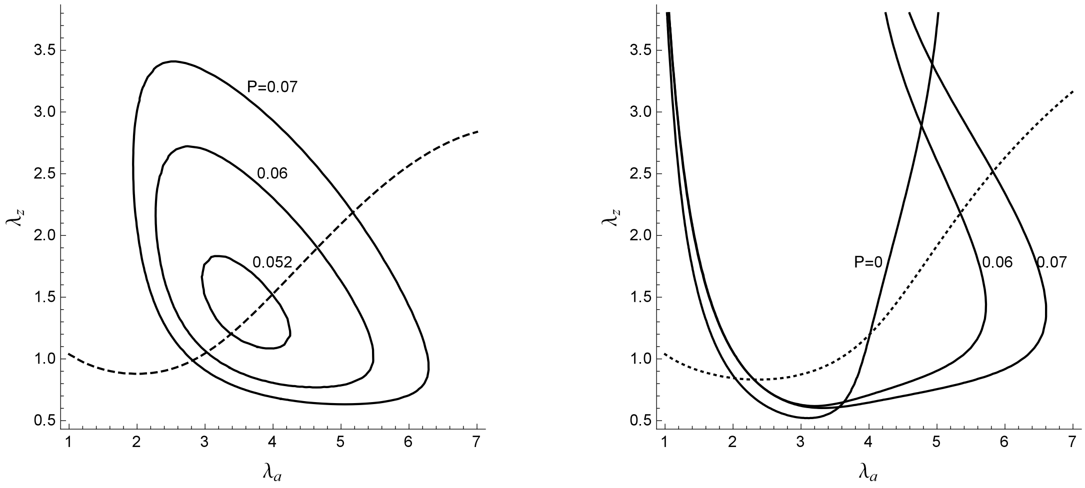
We have also carried out calculations to find out how the results in Figure 3 depend on A. It is found that as A decreases (so that the wall thickness increases), the curve corresponding to $F = 0$ would shift downwards relative to the curve of $\Omega_u(\lambda_a, \lambda_z) = 0$. When A reaches 0.428 approximately, the two curves would no longer intersect, implying that localized bulging becomes impossible below this threshold value. In Table 1, we have listed the critical values of Γ for a selection of values of A. Since the wall thickness is a decreasing function of A/B, it can be seen that the larger the wall thickness, the larger the critical value of Γ. The numbers in brackets in the last row are the corresponding results based on the thin-wall approximation (3-21) and (3-22), and is seen to provide a good approximation, with a relative error of less than 5%, for values of A as small as 0.5.

Table 1: Critical values of ω in free rotation ($F = 0$)

A/B	0.43	0.5	0.6	0.7	0.8	0.9	0.99
λ_a	3.4779	2.9635	2.6486	2.4518	2.3128	2.2085	2.1346
λ_z	0.6291	0.6591	0.6729	0.6797	0.6834	0.6851	0.6956
Γ	1.4446	1.2984	1.1304	0.9958	0.8855	0.7935	0.7230
$\Gamma \approx$	(1.3641)	(1.2397)	(1.0896)	(0.9907)	(0.8837)	(0.7931)	(0.7230)

We next consider the effect of allowing for a non-zero internal pressure. There are now two subcases. The first subcase is when the unconstrained tube is mainly subjected to the action of internal inflation and an axial force, with rotation being small and playing a minor role. This subcase covers the case of zero rotation which has previously been investigated in Fu *et al* (2016). In Figure 4 we have shown the solution of the bifurcation condition $\Omega_u(\lambda_a, \lambda_z) = 0$ for three representative values of Γ and the common value of $A = 0.8$. We note that when Γ is non-zero, the solution again has two branches when the Ogden material model is used. We shall focus our discussion on the parameter regime near $\lambda_a = \lambda_z = 1$. It is seen that adding a rotation to the tube delays the onset of localized bulging when the Gent material is used, but when the Ogden material is used the rotation has a delaying effect when the rotation speed is small, but has an opposite effect when the rotation speed is large enough.

Finally, we assume that the internal pressure P is zero or small, and it is the rotation and axial force F that play a dominant role. In Figure 5 we have shown the solution of $\Omega_u(\lambda_a, \lambda_z) = 0$ for three representative values of P. There is here a big difference between the predictions of the two material models. When $P = 0$ and the Gent material model is used, the condition $\Omega_u(\lambda_a, \lambda_z) = 0$ does not have a solution at all, which means that localized bulging is impossible now matter how large the rotation speed is. Localized bulging first becomes possible when P is increased to the value of 0.051. In contrast, according to the Ogden material model, localized bulging is possible even if P is zero. The dotted line in each figure represents the loading path, the solution of $\tilde{F}(\lambda_a, \lambda_z) = 0$, when $P = 0.06$ and the net axial force is zero.



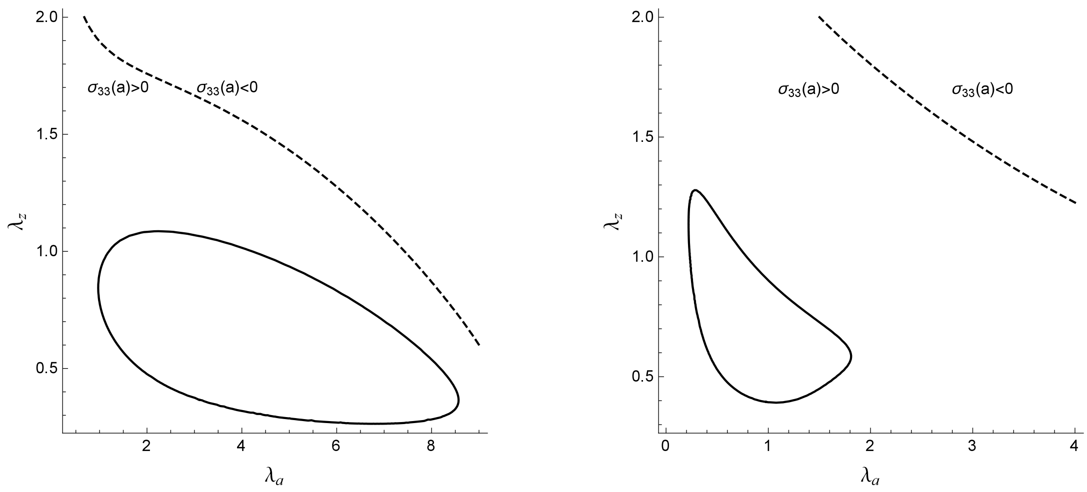
(a) Gent material (b) Ogden material

FIGURE 5. Solution of the bifurcation condition $\Omega_u(\lambda_a, \lambda_z) = 0$ for the three different values of P indicated. The dashed line corresponds to the solution of $\tilde{F}(\lambda_a, \lambda_z) = 0$ when $P = 0.06$, and is the loading path when there is no net axial force applied at the plane end-faces. Its intersection with the solid curve associated with $P = 0.06$ gives the values of the two stretches when localized bulging takes place.

4C. **The shrink-fit case.** In this case the stretches must satisfy $\tilde{F}_0(\lambda_a, \lambda_z, \Gamma) = F$. This equation may also be solved to express Γ in terms of F. The bifurcation condition (2-14) then depends on F as well as λ_a and λ_z. For each specified F, this condition defines a curve in the (λ_a, λ_z) plane. A typical solution with $F = 2$ is shown in Figure 6 where the solution of $\sigma_{33}(a) = 0$ is also shown in a dashed line. It is seen that although localized bulging is possible, the contact force at $r = a$ must necessarily be tensile. This means that no matter how tight the initial fitting is, localized bulging will not occur before the tube loses contact with the rigid spindle. It is also found that as F is gradually reduced, the closed curve shrinks in size and eventually disappear at $F = 1.29$ when the Gent material model is used and at $F = 1.66$ when the Ogden material model is used. Thus, in particular, when the plane ends of the tube are traction-free, localized bulging will not occur no matter how large λ_a or Γ is.

5. CONCLUSION

In this paper we have studied the bifurcation condition for localized bulging of a circular solid cylinder or cylindrical tube which is rotated about its axis of symmetry. Additional external forces may include a non-zero net axial force and/or internal pressure. In each case, the bifurcation condition is derived with the aid of the dynamical systems theory and found to have simple interpretations in terms of physical quantities such as the rotation speed, axial force and internal pressure. For instance, when the axial force and internal pressure are both fixed, the bifurcation condition simply corresponds to the angular velocity reaching a maximum when



(a) Gent material (b) Ogden material

FIGURE 6. Solution of the bifurcation condition (2-14). The result corresponds to $F = 2$ and is shown in solid line. The dashed line in each plot represents the solution of $\sigma_{33}(a) = 0$.

viewed as a function of λ_a for a tube or λ_z for a solid cylinder. However, this correspondence is lost when for instance it is the axial stretch λ_z that is fixed in rotating the tube. Although we have only presented numerical results for $A = 0.8$ and two typical strain-energy functions, our simple representation of the bifurcation condition can be used to evaluate the effects of wall thickness and dependence on the material model in a straightforward manner if required.

The simplest case is perhaps the case of free rotation when both F and P are zero. Our illustrative calculations show that the Ogden material model would predict that localized bulging may occur but the Gent material model would predict that localized bulging can never occur. According to the Gent material model, localized bulging could occur only if a sufficiently large internal pressure is added. Also, according to both models, localized bulging will not occur in the shrink-fit case before contact with the rigid spindle is lost.

It would be of interest to ask whether the localized bulging discussed in the current study could be realized for realistic material parameters and angular speeds in free rotation. As a rough estimate, we take

$$\rho = 0.91\text{kg/m}^3, \quad \mu = 667\text{kPa},$$

which are typical values for natural rubber. If $B = 0.1$ m, the critical value of $\Gamma = 3.81$ predicted by the Ogden model for a solid cylinder would correspond to an angular velocity $\omega = 528/\text{s}$, which translates to a rotation frequency of 5046 rpm (revolutions per minute). For a cylindrical tube with $B = 0.1$ m, $A/B = 0.8$, the critical value of $\Gamma = 0.885$ predicted by the Ogden model for an unconstrained tube corresponds to an angular velocity $\omega = 270/\text{s}$, which is about 2582 rpm. This is within the achievable rotor speed of 3000 rpm. The required speed for the solid cylinder is higher, but it can be brought down by choosing a softer material since the rotation frequency is proportional to the square root of μ. For instance, for a typical Gelatin, the shear modulus can be as small as 30 kPa (Markidou *et al* 2005) for which the critical rotation frequency would be 1070 rpm. Thus, we may conclude that localized bulging should be easily observable in soft materials.

In the current study, we have assumed that the solid cylinder or tube is made of an isotropic elastic material. Our study can be easily extended to the case when the tube is helically reinforced by two families of identical fibres. The symmetry of the fibres ensures that axi-symmetric deformations are again possible, and as far as such deformations are concerned the tube would behave effectively like an isotropic material. However, as demonstrated in a recent study for the pure inflation case (Wang and Fu 2017), it is expected that the fibres will have a drastic effect on localised bulging. To be more precise, using fibre-reinforcement can be a very effective method to construct anti-bulging tubes.

Finally, we remark that the bifurcation condition derived in the present paper is only a necessary condition for localized bulging to occur. Strictly speaking, whether localized bulging can actually take place at the critical value of rotation can only be established by a weakly nonlinear analysis. However, from our experience with the pure inflation case for which both analytical and experimental results are available, we expect that the existence of localized bulging configurations should be the norm rather than the exception.

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