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Corresponding Author: Prof. Y. Fu,

Corresponding Author's Institution: Keele University

First Author: Y. Fu

Order of Authors: Y. Fu; Yanqi Song, PhD

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A note on perturbation formulae for the surface-wave speed due to perturbations in material properties

Y.Q. Song[1] and Y.B. Fu[2]

[1]*Department of Engineering Mechanics, China University of Mining and Technology, Beijing, China.*

[2]*Department of Mathematics, Keele University, Staffordshire ST5 5BG, UK*

Abstract

In a recent paper by Tanuma and Man, a two-term asymptotic formula was derived for the speed of surface waves propagating in an anisotropic elastic half-space whose elastic moduli differ only slightly from those for a (base) isotropic elastic material. This formula disagrees with that derived by Delsanto and Clark in an earlier paper using a different method. In this short note, we use a simple procedure to derive another two-term asymptotic formula for the surface-wave speed. Our formula takes the same compact form even if the base material is generally anisotropic. We show that when an error in the work of Delsanto and Clark is corrected, the three different methods do give equivalent results.

Keywords: Surface waves; Stroh formulation; Surface-wave speed.

1 Some results from the surface-wave theory

Surface-wave theory is known to have a wide range of applications. In particular, there are situations, such as acoustoelasticity and design of surface acoustic wave (SAW) devices, in which a small perturbation in the surface-wave speed needs to be determined when material properties are slightly perturbed. We refer to Delsanto and Clark [1] and Tanuma and Man [2, 3] for a comprehensive review of the literature and for a description of the background for the problem addressed in the present paper. In this section, we collect together some results from the surface-wave theory that will be used later.

We consider surface waves propagating in a generally anisotropic elastic half-space. We choose a rectangular coordinate system so that the interior of the half-space is defined by $0 < x_2 < \infty$ and the surface waves propagate along the x_1-axis. The surface-wave speed is obtained by solving the equation of motion

$$c_{ijkl}u_{k,lj} = \rho \ddot{u}_i, \quad 0 < x_2 < \infty \tag{1.1}$$

subject to the traction-free boundary condition

$$c_{i2kl}u_{k,l} = 0 \quad \text{on} \;\; x_2 = 0, \tag{1.2}$$

and the decay condition $u_i \to 0$ as $x_2 \to \infty$, where we have used the standard notation: c_{ijkl} are the elastic moduli, u_i are the displacement components, ρ is the density, repeated suffices are summed from 1 to 3, a comma denotes partial differentiation with respect to a spatial coordinate, and a superimposed dot denotes partial differentiation with respect to the time. Throughout this note we only impose the symmetry $c_{ijkl} = c_{klij}$ and allow for the possibility $c_{ijkl} \neq c_{jikl}$ so that our results can easily be adapted to surface-wave propagation in a pre-stressed elastic material or a material with residual stress.

There is a number of ways to construct the surface-wave solution. One way is to start with a solution of the form

$$\boldsymbol{u} = \boldsymbol{z}(x_2)\, \mathrm{e}^{\mathrm{i}(x_1 - vt)}, \quad \boldsymbol{z}(x_2) = \mathrm{e}^{-Ex_2}\boldsymbol{z}(0), \tag{1.3}$$

where v is the surface-wave speed, the matrix E and vector function $\boldsymbol{z}(x_2)$ are to be determined, and without loss of generality we have taken the wave number in (1.3) to be unity (since the present problem does not have a natural lengthscale). On substituting $(1.3)_1$ into (1.1) and (1.2), we obtain

$$T\boldsymbol{z}''(x_2) + \mathrm{i}(R + R^T)\boldsymbol{z}'(x_2) - (Q - \rho v^2 I)\boldsymbol{z}(x_2) = \boldsymbol{0}, \quad 0 < x_2 < \infty \tag{1.4}$$

$$\boldsymbol{t}(x_2) \equiv -T\boldsymbol{z}'(x_2) - \mathrm{i}R^T\boldsymbol{z}(x_2) = \boldsymbol{0}, \quad \text{on } x_2 = 0, \tag{1.5}$$

where the first relation in (1.5) defines the vector $\boldsymbol{t}$ and the matrices T, R, Q are defined by their components

$$T_{ij} = c_{i2j2}, \quad R_{ij} = c_{i1j2}, \quad Q_{ij} = c_{i1j1}. \tag{1.6}$$

A useful quantity in the surface-wave theory is the surface-impedance matrix $M(v)$ defined by

$$\boldsymbol{t}(x_2) = M(v)\boldsymbol{z}(x_2). \tag{1.7}$$

We observe that $M(v)$ is independent of x_2 when the material is homogeneous [4]. For a free-surface wave, we obtain from the boundary condition (1.5) the secular equation $\det M(v) = 0$ that determines the surface-wave speed [5]. In the case when the moduli c_{ijkl} are the classical elastic moduli for a generally anisotropic, unstressed elastic material that satisfy the strong convexity condition, it is now well-known [6, 7] that this secular equation should usually have a unique solution in the subsonic interval $0 < v < \hat{v}$, where $\hat{v}$ is known as the limiting speed [6]. For a prestressed material or a material with residual stress, the strong convexity condition is no longer appropriate. We then impose the conditions that the moduli c_{ijkl} satisfy the strong ellipticity condition and are such that the surface-impedance matrix $M(v)$ is positive definite when $v = 0$. It can be gleaned from the analysis of Fu and Mielke [8] that under these assumptions, the secular equation $\det M(v) = 0$ should again have a unique root in the subsonic interval with the limiting speed $\hat{v}$ similarly defined.

With the use of (1.4) and $(1.5)_1$, the first-order derivatives $\boldsymbol{z}'(x_2)$ and $\boldsymbol{t}'(x_2)$ can easily be written as a linear combination of $\boldsymbol{z}(x_2)$ and $\boldsymbol{t}(x_2)$, thus leading to the Stroh formulation

$$\begin{pmatrix} \boldsymbol{z}'(x_2) \\ \mathrm{i}\,\boldsymbol{t}'(x_2) \end{pmatrix} = \mathrm{i}N \begin{pmatrix} \boldsymbol{z}(x_2) \\ \mathrm{i}\,\boldsymbol{t}(x_2) \end{pmatrix}, \quad N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_1^T \end{pmatrix}, \tag{1.8}$$

where $\mathrm{i} = \sqrt{-1}$ and

$$N_1 = -T^{-1}R^T, \quad N_2 = T^{-1}, \quad N_3 = RT^{-1}R^T - Q + \rho v^2 I.$$

The above Stroh formulation owes its success to the fact that the traction vector and the displacement are a pair of (work) conjugate variables.

By substituting (1.7) into (1.8), we find that the matrix E in (1.3) and the surface-impedance matrix are related by

$$E = T^{-1}(M + \mathrm{i}R^T), \tag{1.9}$$

and that $M(v)$ satisfies the matrix equation [8, 9, 10, 11]

$$(M - \mathrm{i}R)T^{-1}(M + \mathrm{i}R^T) - Q + \rho v^2 I = 0. \tag{1.10}$$

This matrix equation does not have a unique solution, but the solution relevant to our surface-wave solution must be such that the three eigenvalues of E, computed from (1.9), all have positive real parts [8]. It is shown in [12] that in the subsonic interval such a solution is unique. We also observe that if p_1, p_2, p_3 denote the three eigenvalues of N that have positive imaginary parts, then the three eigenvalues of E are $-\mathrm{i}p_1, -\mathrm{i}p_2, -\mathrm{i}p_3$. In fact, the three matrices N, M and E are related by

$$N \begin{pmatrix} I \\ \mathrm{i}M \end{pmatrix} = \mathrm{i} \begin{pmatrix} I \\ \mathrm{i}M \end{pmatrix} E; \tag{1.11}$$

see Mielke and Fu [12].

2 Perturbation formula

We now assume that the material considered in the previous section is obtained by slightly perturbing properties of another base material that is characterized by the elastic moduli $c^{(0)}_{ijkl}$. Thus, we may write

$$c_{ijkl} = c^{(0)}_{ijkl} + \epsilon c^{(1)}_{ijkl}, \tag{2.1}$$

where ϵ is a small positive parameter introduced to indicate explicitly that the perturbations $\epsilon c^{(1)}_{ijkl}$ are of small amplitude. In view of (2.1), we have

$$T = T_0 + \epsilon T_1, \quad R = R_0 + \epsilon R_1, \quad Q = Q_0 + \epsilon Q_1, \tag{2.2}$$

where the two groups of matrices (T_0, R_0, Q_0) and (T_1, R_1, Q_1) are counterparts of (T, R, Q) constructed from the moduli $c^{(0)}_{ijkl}$ and $c^{(1)}_{ijkl}$, respectively. Corresponding to the above notation, we also have

$$M = M_0 + \epsilon M_1 + O(\epsilon^2), \quad \rho v^2 = X_0 + \epsilon X_1 + O(\epsilon^2), \tag{2.3}$$

where X_0 is the value of ρv^2 associated with the base material and our task is to express the correction term X_1 in terms of the scaled perturbations T_1, R_1 and Q_1.

On substituting (2.2) and (2.3) into (1.10), and equating the coefficients of ϵ on the two sides, we obtain

$$E_0^{*T} M_1 + M_1 E_0 = K - X_1 I, \tag{2.4}$$

where

$$E_0 = T_0^{-1}(M_0 + \mathrm{i}R_0^T), \quad K = Q_1 + \mathrm{i}R_1 E_0 - \mathrm{i}E_0^{*T} R_1^T + E_0^{*T} T_1 E_0, \tag{2.5}$$

and a superscript "*" denotes complex conjugation. The matrix equation (2.4) is recognized as a Liapunov matrix equation (see, e.g., [13, p. 307]), and under conditions satisfied by our matrix E_0 it has a unique solution given by

$$M_1 = \int_0^\infty \mathrm{e}^{-x_2 E_0^{*T}} (K - X_1 I) \mathrm{e}^{-x_2 E_0} dx_2. \tag{2.6}$$

For the base and perturbed materials, the boundary condition (1.5) takes the form

$$M_0 \boldsymbol{d}_0 = \boldsymbol{0}, \quad M\boldsymbol{d} = \boldsymbol{0}, \tag{2.7}$$

where $\boldsymbol{d}_0 = \boldsymbol{z}_0(0)$, $\boldsymbol{d} = \boldsymbol{z}(0)$ and $\boldsymbol{d} = \boldsymbol{d}_0 + O(\epsilon)$. Contracting $(2.7)_2$ with $\boldsymbol{d}_0^*$ and rearranging the expression, we obtain

$$\begin{aligned} \boldsymbol{d}_0^* \cdot M\boldsymbol{d} &= \boldsymbol{d} \cdot (M_0^T + \epsilon M_1^T)\boldsymbol{d}_0^* = \boldsymbol{d} \cdot (M_0^* + \epsilon M_1^*)\boldsymbol{d}_0^* \\ &= \boldsymbol{d} \cdot \epsilon M_1^* \boldsymbol{d}_0^* = \epsilon \boldsymbol{d}_0 \cdot M_1^* \boldsymbol{d}_0^* = \epsilon \boldsymbol{d}_0^* \cdot M_1 \boldsymbol{d}_0 = 0, \end{aligned}$$

where we have consistently neglected the $O(\epsilon^2)$ terms and have made use of the fact that M, M_0 and M_1 are all Hermitian. Thus, we obtain $\boldsymbol{d}_0^* \cdot M_1 \boldsymbol{d}_0 = 0$, which then yields, with the use of (2.6),

$$X_1 \int_0^\infty \mathrm{e}^{-x_2 E_0^*} \boldsymbol{d}_0^* \cdot \mathrm{e}^{-x_2 E_0} \boldsymbol{d}_0 dx_2 = \int_0^\infty \mathrm{e}^{-x_2 E_0^*} \boldsymbol{d}_0^* \cdot K \mathrm{e}^{-x_2 E_0} \boldsymbol{d}_0 dx_2. \tag{2.8}$$

We denote $\mathrm{e}^{-x_2 E_0} \boldsymbol{d}_0$ in the above expression by $\boldsymbol{z}_0(x_2)$. We note that $\boldsymbol{z}_0$ is simply the counterpart of $\boldsymbol{z}(x_2)$ for the base material and we have $\boldsymbol{z}_0' = -E_0 \boldsymbol{z}_0$. With the use of these facts, it can easily be shown that (2.8) may be rewritten as

$$X_1 \int_0^\infty |\boldsymbol{z}_0|^2 dx_2 = \int_0^\infty \left\{ \boldsymbol{z}_0^* \cdot Q_1 \boldsymbol{z}_0 - \mathrm{i}\boldsymbol{z}_0^* \cdot R_1 \boldsymbol{z}_0' + \mathrm{i}\boldsymbol{z}_0 \cdot R_1 \boldsymbol{z}_0^{*'} + \boldsymbol{z}_0^{*'} \cdot T_1 \boldsymbol{z}_0' \right\} dx_2. \tag{2.9}$$

To facilitate evaluation of the integrals above, we assume that the m-th component of $\boldsymbol{z}_0(x_2)$ is written in the form

$$(\boldsymbol{z}_0)_m = b_{mn}\mathrm{e}^{\mathrm{i}p_n x_2}, \tag{2.10}$$

where b_{mn} and p_n are known constants (in particular, p_1, p_2, p_3 are the same as those appearing at the end of the previous section). For instance, when the base material is isotropic, we have

$$p_1 = \mathrm{i}\sqrt{1-\frac{\rho v_0^2}{\mu}}, \quad p_2 = \mathrm{i}\sqrt{1-\frac{\rho v_0^2}{\lambda+2\mu}}, \tag{2.11}$$

$$b_{11} = -2p_1p_2, \quad b_{12} = p_1^2 - 1, \quad b_{21} = 2p_2, \quad b_{22} = p_2(p_1^2-1), \tag{2.12}$$

and $b_{3j} = b_{j3} = 0$ $(j = 1, 2, 3)$, where λ and μ are the usual Lamé constants.

On substituting (2.10) into (2.9) and evaluating the integrals, we obtain

$$X_1 = I_2/I_1, \tag{2.13}$$

where

$$I_1 = \frac{b_{im}b_{ik}^*}{p_m - p_k^*}, \quad I_2 = \frac{b_{ik}^* Q_{ij}^{(1)} b_{jm}}{p_m - p_k^*} + \frac{b_{ik}^* p_k^* T_{ij}^{(1)} b_{jm} p_m}{p_m - p_k^*}$$
$$+\frac{b_{ik}^* b_{jm} p_m R_{ij}^{(1)}}{p_m - p_k^*} + \frac{b_{im} b_{jk}^* p_k^* R_{ij}^{(1)}}{p_m - p_k^*}, \tag{2.14}$$

where we have written $T^{(1)}, R^{(1)}, Q^{(1)}$ for T_1, R_1, Q_1, and the subscripts i, j, m, k are all summed from 1 to 3.

Equation (2.13) together with (2.14) provides a compact formula for computing the surface-wave perturbation when the material properties are slightly perturbed, and it is valid even if the base material is generally anisotropic. We observe that for surface waves polarized in a symmetry plane of a monoclinic material, we may set $b_{3j} = b_{j3} = 0$ $(j = 1, 2, 3)$ and the above formula then shows that X_1 will not involve any modulus $c_{ijkl}^{(1)}$ in which suffix 3 appears at least once. This fact was first noted by Delsanto and Clark [1] when the base material is isotropic.

3 Comparison

When the base material is isotropic, which is the case studied by Delsanto and Clark [1] and Tanuma and Man [3], we have found that the expression (2.13) together with (2.11), (2.12) and (2.14) is equivalent to the formula given by Tanuma and Man [3] (in establishing the equivalence we needed to use the cubic equation for the surface-wave speed to eliminate cubic and higher powers of the speed). In order to resolve the discrepancy between Tanuma and Man's [3] formula and the corresponding formula given by Delsanto and Clark [1], we repeated Delsanto and Clark's calculations. We found that their equation (39) is in error and its right hand side should be replaced by

$$W_1^{(2)}\left(B_2^{(1)'}+\frac{u-2}{2\sqrt{y_1}}B_3^{(1)'}\right)-W_1^{(1)}\left(B_2^{(2)'}+\frac{u-2}{2\sqrt{y_2}}B_3^{(2)'}\right).$$

With this error identified, we were able to show that Delsanto and Clark's method, although laborious, did give an equivalent result to that of ours and Tanuma and Man's. Delsanto and Clark's formula (41) is correct except that the A_{2223} and A_{3332} in this expression should be identically zero.

We conclude by remarking that another elementary method that can be used for the present purpose is by looking for a regular perturbation solution of the form $\boldsymbol{u}=\boldsymbol{u}^{(0)}+\epsilon\boldsymbol{u}^{(1)}+\cdots$ and then imposing a solvability condition on the $O(\epsilon)$ problem satisfied by $\boldsymbol{u}^{(1)}$ and X_1. This is of course a routine procedure and various forms of this method have previously been implemented by Sinha *et al* [14], Fu [15] and Norris [16] for a variety of elasticity problems. We have checked to confirm that this method also gives the same expression (2.9).

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References

1. Delsanto, P.P., Clark Jr, A.V.: Rayleigh wave propagation in deformed orthotropic materials. J. Acoust. Soc. Am. **81**, 952-960 (1987).

2. Tanuma, K., Man, C.-S.: Angular dependence of Rayleigh-wave velocity in prestressed polycrysstalline media with monoclinic texture. J. Elast. **69**, 181-214 (2002).

3. Tanuma, K., Man, C.-S.: Perturbation formula for phase velocity of Rayleigh waves in prestressed anisotropic media. J. Elast. **85**, 21-37 (2006).

4. Ingebrigtsen, K.A., Tonning, A.: Elastic surface waves in crystals. Phys. Rev. **184**, 942-951 (1969).

5. Stroh, A.N.: Steady state problems in anisotropic elasticity. J. Math. Phys. **41**, 77-103 (1962).

6. Chadwick, P., Smith, G.D.: Foundation of the theory of surface waves in anisotropic elastic materials. Adv. Appl. Mech. **17**, 303-376 (1977).

7. Barnett, D.M., Lothe, J.: Free surface (Rayleigh) waves in anisotropic elastic half-spaces: the surface impedance method. Proc. Roy. Soc. Lond. A**402**, 135-152 (1985).

8. Fu, Y.B., Mielke, A.: A new identity for the surface-impedance matrix and its application to the determination of surface-wave speeds. Proc. Roy. Soc. Lond. A**458**, 2523-2543 (2002).

9. Biryukov, S.V.: Impedance method in the theory of elastic surface waves. Sov. Phys. Acoust. **31**, 350-354 (1985).

10. Mielke, A., Sprenger, P.: Quasiconvexity at the boundary and a simple variational formulation of Agmon's condition. J. Elast. **51**, 23-41 (1998).

11. Fu, Y.B.: An explicit expression for the surface-impedance matrix of a generally anisotropic incompressible elastic material in a state of plane strain. Int. J. Non-linear Mech. **40**, 229-239 (2005).

12. Mielke, A., Fu, Y.B.: Uniqueness of the surface-wave speed: A proof that is independent of the Stroh formalism. Mathematics and Mechanics of Solids **9**, 5-15 (2004).

13. Barnett, S.: Matrices: methods and applications, Oxford University Press (1992).

14. Sinha, B.K., Norris, A.N., Chang, S.-K.: Borehole flexural modes in anisotropic formations. Geophysics **59**, 1037-1052 (1994).

15. Fu, Y.B.: Resonant-triad instability of a pre-stressed elastic plate. J. Elast. **41**, 13-37 (1995).

16. Norris, A.N.: Small-on-large theory with applications to granular materials and fluid/solid systems. In: Destrade, M, Sacommandi, G. (eds.) Waves in Nonlinear Pre-Stressed Materials, CISM Courses and Lectures, to appear.

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