

A semi-analytical approach to Biot instability **in a growing layer**: strain gradient correction, weakly nonlinear analysis and imperfection sensitivity

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Abstract

Many experimental works have recently investigated the dynamics of crease formation during the swelling of long soft slabs attached to a rigid substrate. Mechanically, the spatially constrained growth provokes a residual strain distribution inside the material, and therefore the problem is equivalent to the uniaxial compression of

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an elastic layer.

The aim of this work is to propose a semi-analytical approach to study the non-linear buckling behaviour of a growing soft layer. We consider the presence of a microstructural length, which describes the **effect of a simple strain gradient correction in the growing hyperelastic layer, considered as a neo-Hookean material**. By introducing a nonlinear stream function for enforcing exactly the incompressibility constraint, we develop a variational formulation for performing a stability analysis of the basic homogeneous solution. At the linear order, we derive the corresponding dispersion relation, proving that even a small strain gradient effect allows the system to select a critical dimensionless wavenumber while giving a small correction to the Biot instability threshold. A weakly nonlinear analysis is then performed by applying a multiple-scale expansion to the neutrally stable mode. By applying the global conservation of the mechanical energy, we derive the Ginzburg-Landau equation for the critical single mode, identifying a pitchfork bifurcation. Since the bifurcation is found to be subcritical for a small ratio between the microstructural length and the layer's thickness, we finally perform a sensitivity analysis to study the effect of the initial presence of a sinusoidal imperfection on the free surface of the layer. In this case, the incremental solution for the stream function is written as a Fourier series, so that the surface imperfection can have a cubic resonance with the linear modes. The solutions indicate the presence of a turning point close to the critical threshold for the perfect system. We also find that the inclusion of higher modes has a steepening effect on the surface profile, indicating the incipient formation of an elastic singularity, possibly a crease.

Keywords: Buckling, Biot instability, growth, weakly nonlinear analysis, imperfection sensitivity

1 Introduction

Modelling the buckling behaviour of a compressed elastic layer has attracted much interest in the mechanics community in decades. Although the linear stability analysis dates back to the seminal work of Biot [4] for a homogeneous elastic half-space, a theoretical understanding of the fully nonlinear characteristics has completely eluded researchers for a long time. In fact, this problem lacks a typical length-scale, and thus all modes are found to be equally unstable at the linear compression threshold. Accordingly, a second-order interaction of all the Fourier modes must be accounted in a post-buckling analysis, which unfortunately result in a series of solvability conditions not converging to a non-trivial solution. This peculiar behaviour has suggested that a non-trivial solution could occur only in presence of a discontinuity, which could be identified as a static shock or a crease [24].

Many experimental works have been performed to describe the dynamics of crease formation during the swelling of polymeric gels attached to a rigid substrate [26, 27, 21]. In fact, this is identified as a simple system model both for guiding fabrication of controllable surfaces in many biomedical applications [20, 5] and for investigating pattern formation in growing living matter [22, 11]. Mechanically, the spatially constrained growth provokes a residual strain distribution inside the material, and therefore the problem is equivalent to the uniaxial compression of an elastic layer. Although the thickness of the layer introduces a typical length-scale, the first unstable mode at the linear stability threshold is found having zero wavelength, thus recovering the simple result that the free surface of the layer locally behaves as an elastic half-space.

The presence of any microstructural effect, such as a surface tension, no matter how small, can regularize the results of the linear stability analysis, and a finite critical mode can be identified [2]. Therefore, a natural choice in subsequent studies has been to investigate numerically the buckling behaviour taking the limit of such a surface effect going to zero [18]. Creasing has been found to be a fully nonlinear instability, like typical nucleation

phenomena, which remains undetectable by linear stability methods [19]. Nonetheless, creasing cannot occur before the Biot limit unless the system symmetry is broken by hand. Very little is known about the nonlinear behaviour in this regime, which is the main goal of this contribution. In an accompanying paper, we have proposed a semi-analytical approach to this problem by taking the limit from a coated to a homogeneous half-space [16]. Moreover, a similar approach has been also applied to equi-biaxial compression, demonstrating that the emergence of hexagonal patterns is energetically favoured [12]. In this work, we extend our theoretical analysis by considering the presence of a microstructural length which describes the **effect of a simple strain gradient correction in the growing hyperelastic layer, considered as a neo-Hookean material**.

The paper is organized as follows. In Section 2, we describe the theoretical framework for the elastic growth model and we derive the basic homogeneous solution for planar deformations. In Section 3, we propose a variational formulation for solving the incremental elastic problem using a nonlinear stream function, which takes into account exactly for the incompressibility constraint. In Section 4, we perform a weakly nonlinear stability analysis using the perturbation method of multiple-scale development, whilst in Section 5 we derive the amplitude equations for the critical mode using an energetic approach. Moreover, in Section 6 we perform a sensitivity analysis to imperfections considering the weakly nonlinear behaviour of an initial layer with surface undulations. The results of this work are finally summarized in Section 7, together with some concluding remarks.

2 The elastic growth model and its basic solution

Let us consider an elastic layer with a vertical thickness H much smaller than the horizontal lengths. Indicating with $\mathbf{X}(X, Y, Z)$ (resp. $\mathbf{x}(x, y, z)$) its material (resp. spatial) position vector, the geometrical deformation tensor is defined as $\mathbf{F} = \text{Grad } \mathbf{x}$. As proposed by Rodriguez et al. [25], the volumetric growth can be modeled by a multiplicative

decomposition of \mathbf{F} , as follows:

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_g, \quad (1)$$

where \mathbf{F}_e is the elastic deformation tensor and the growth tensor \mathbf{F}_g is a linear map of the tangent space defining a natural grown state. Whenever \mathbf{F}_g is not compatible with the spatial constraints, the elastic deformation restores the overall compatibility of the layer's deformation, whilst residual stresses arise. For the sake of simplicity, we consider a homogeneous, isotropic growth tensor in the form $\mathbf{F}_g = g\mathbf{I}$, where g is the isotropic growth rate and \mathbf{I} is the unit tensor. Furthermore, we undertake in the following a plane strain assumption, considering a geometrical constraint only in the horizontal direction X while neglecting any functional dependence on Y .

From a constitutive viewpoint, the body is modeled as a strain gradient material, so that its strain energy density W reads

$$W = \frac{\mu}{2} (\text{tr } \mathbf{b}_e - 3) + \frac{K}{2} \text{tr} ((\text{Grad } \mathbf{F}_e)^T : \text{Grad } \mathbf{F}_e), \quad (2)$$

where μ is the shear modulus of the neo-Hookean matrix, $\mathbf{b}_e = \mathbf{F}_e \mathbf{F}_e^T$ is the elastic left Cauchy-Green deformation tensor, and K is a strain gradient elastic coefficient [14]. In particular, we can define $\ell = \sqrt{K/\mu}$ as a characteristic microstructural length which defines the typical scale of the non-local elastic effects.

The basic elastic solution for a growing layer, whose fields are indicated with superscript (0) in the following, corresponds to a homogeneous elastic deformation $\mathbf{F}_e^{(0)} = \text{diag}(\lambda_x, 1, \lambda_z)$ whose principal stretches read $\lambda_x = 1/g$ and $\lambda_z = g$, restoring the compatibility with the constraint of zero displacement at the layer's bottom $Z = 0$. Using the constitutive assumptions in Eq.(2), we find that the residual Cauchy stress $\boldsymbol{\sigma}^{(0)}$ for the layer is given by:

$$\boldsymbol{\sigma}^{(0)} = \mu \mathbf{b}_e^{(0)} - p_0 \mathbf{I}, \quad (3)$$

where p_0 is a Lagrange multiplier due to the constraint of incompressibility, i.e. $\det \mathbf{F}_e = 1$, which can be fixed through the stress free boundary conditions at $Z = H$, as $p_0 = \lambda_z^2 \mu$.

In the following section, we perform a stability analysis of the elastic problem using a variational formulation based on the definition of a nonlinear stream function.

3 A variational approach of the elastic problem

3.1 Canonical transformation using a stream function

Dealing with an incompressibility constraint in plane elasticity, the application of a canonical transformation allows us to write the boundary value problem in terms of a stream function $\psi(x, Z)$ [9, 10]. Accordingly, let us take into account the following solution:

$$z = \psi_{,x}; \quad X = \psi_{,Z}/g^2, \quad (4)$$

where comma denotes partial differentiation. In particular, considering a mixed coordinate state (x, Z) , the elastic deformation gradient can be decomposed as $\mathbf{F}_e = \mathbf{F}_a \mathbf{F}_b$, with:

$$\mathbf{F}_a = \begin{bmatrix} 1 & 0 \\ \psi_{,xx} & \psi_{,xZ} \end{bmatrix}; \quad \mathbf{F}_b = \begin{bmatrix} g/\psi_{,xZ} & -\psi_{,ZZ}/(g\psi_{,xZ}) \\ 0 & 1/g \end{bmatrix} \quad (5)$$

so that:

$$\mathbf{F}_e = \begin{bmatrix} \frac{g}{\psi_{,xZ}} & -\frac{\psi_{,ZZ}}{g\psi_{,xZ}} \\ g\frac{\psi_{,xx}}{\psi_{,xZ}} & \frac{\psi_{,xZ}}{g} - \frac{\psi_{,xx}\psi_{,ZZ}}{g\psi_{,xZ}} \end{bmatrix}, \quad (6)$$

which identically satisfies the constraint $\det \mathbf{F}_e = 1$. Using the combination laws of second derivatives [13], the material gradient of the deformation tensor can be written as:

$$\text{Grad } \mathbf{F}_e = \text{Grad } \mathbf{F}_a : [\mathbf{F}_b, \mathbf{F}_b] - \mathbf{F}_g \text{Grad } (\mathbf{F}_b^{-1}) : [\mathbf{F}_b, \mathbf{F}_b] \quad (7)$$

where Grad is the gradient with respect to the mixed coordinate state, and we used the short-hand notation for a product rule for third-order tensors as $(\mathbb{C} : [A, B])_{ijk} = \mathbb{C}_{i\alpha\beta} A_{\alpha j} B_{\beta k}$.

In summary, the Lagrangian density \mathcal{L} of the continuum can be expressed as a function

of the stream function, being:

$$\mathcal{L} = (K - W) = \frac{1}{2}\rho\mathbf{v}^2 - W \quad (8)$$

where ρ is the material density and \mathbf{v} is the spatial velocity. Recalling that $\frac{dX}{dt} = \frac{dZ}{dt} = 0$, the kinetic energy density K can be rewritten in mixed coordinates as:

$$K = \frac{1}{2}\rho \left[\left(\frac{\psi_{,tZ}}{\psi_{,xZ}} \right)^2 + \left(\psi_{,tx} - \frac{\psi_{,tZ}\psi_{,xx}}{\psi_{,xZ}} \right)^2 \right] \quad (9)$$

so that the Lagrangian density is a function of the partial derivatives of the strain function. In the following, we derive the Euler-Lagrange equations of the incremental elastic boundary value problem. Dealing with a nonlinear stream function, such an approach has the notable advantage of fulfilling exactly the incompressibility constraint, so that no Lagrange multiplier must be introduced.

3.2 Variational formulation and linear stability analysis

A perturbation on the homogeneous elastic solution can be imposed by defining an increment $f(x, Z, t)$ of the elastic stream function, as follows:

$$\psi = g^2 xZ + \epsilon f(x, Z, t) \quad (10)$$

where **without loss of generality and following standard practice, we may assume ϵ is a small positive parameter**. The total Lagrangian energy L of the body can be written as:

$$L = \iiint_V \mathcal{L} dx dy dz = \iiint_{\bar{V}} (\mathcal{L}\psi_{,xZ}) dx dy dZ \quad (11)$$

where \bar{V} is the current body volume expressed in terms of the mixed coordinates. Thus, the incremental problem can be transformed into the variational problem $\delta L = 0$ for a generic perturbation of the stream function. The corresponding bulk Euler-Lagrange equations read:

$$\left(\frac{\partial L}{\partial f_{,\alpha\beta}} \right)_{,\alpha\beta} - \left(\frac{\partial L}{\partial f_{,ijk}} \right)_{,ijk} = 0 \quad (12)$$

where $(\alpha, \beta) = (x, Z, t)$, $(i, j, k) = (x, Z, t)$, and summation on **repeated** subscripts is assumed.

Expressing the energy terms in terms of the stream function, the bulk incremental equation at the linear order takes the following simplified form:

$$\left\{ g^2 \left(\Delta - \frac{\rho}{\mu} \partial_{tt} \right) - \ell^2 \Delta^2 \right\} \bar{\Delta} f = 0 \quad (13)$$

where $\Delta = (\partial_{xx} + \partial_{ZZ})$ and $\bar{\Delta} = (g^4 \partial_{xx} + \partial_{ZZ})$. In particular, we notice that it is a sixth-order partial differential problem, where the higher order derivative are multiplied by the microstructure length ℓ , whose small value therefore indicates the formation of a boundary layer [3]. Furthermore, six boundary Euler-Lagrange equations arise at the boundaries when assuming periodicity on x . Two emerge at the fixed surface $Z = 0$, which read:

$$f_{,x}(x, 0, t) = f_{,Z}(x, 0, t) = 0 \quad (14)$$

The four remaining correspond to vanishing incremental stresses and hyperstresses at the free surface $Z = H$. Since they are given by the surface terms arising by integrating by parts the variational problem, there is not a unique representation, and their particular form is driven by the chosen expression of the perturbation. In case of sinusoidal waves, it is possible to show that the two conditions on the generalized stresses read:

$$\left(\frac{\partial L}{\partial f_{,ZZ}} \right)_{,Z} + \left(\frac{\partial L}{\partial f_{,xZ}} \right)_{,x} + \left(\frac{\partial L}{\partial f_{,tZ}} \right)_{,t} - \left(\frac{\partial L}{\partial f_{,ZZZ}} \right)_{,ZZ} - \left(\frac{\partial L}{\partial f_{,xZZ}} \right)_{,xZ} = 0 \quad \text{at } Z = H \quad (15)$$

$$\left(\frac{\partial L}{\partial f_{,ZZ}} \right) - \left(\frac{\partial L}{\partial f_{,ZZZ}} \right)_{,Z} - \left(\frac{\partial L}{\partial f_{,xZZ}} \right)_{,x} = 0 \quad \text{at } Z = H \quad (16)$$

while the incremental conditions on the hyperstresses are given by:

$$\left(\frac{\partial L}{\partial f_{,ZZZ}} \right) = 0 \quad \text{at } Z = H \quad (17)$$

$$\left(\frac{\partial L}{\partial f_{,xxZ}} \right) = 0 \quad \text{at } Z = H \quad (18)$$

Let us now consider a traveling sinusoidal wave **propagating in the x-direction**, so that

the perturbation reads:

$$f(x, Z, t) = U(Z) \exp[Ik(x - Vt)] + c.c. \quad (19)$$

where I is the imaginary unit, k is the real wavenumber, V is the wave speed and $c.c.$ denotes the complex conjugate. If we take the limit $\ell/H \rightarrow 0$, the incremental elastic problem can be expressed by **the following fourth-order differential equation on $U = U(Z)$:**

$$U'''' - k^2(1 + g^4)U'' + k^4U = 0 \quad (20)$$

where the prime denotes differentiation, and whose solution is given by:

$$U(Z) = \cosh(qkg^2Z) - \cosh(kg^2Z) + B(\sinh(qkg^2Z) - q \sinh(kg^2Z)) \quad (21)$$

with:

$$q = \frac{1}{g^2} \sqrt{1 - \frac{\rho}{\mu} V^2 g^2} \quad (22)$$

In particular, the solution in Eq.(21) already takes into account the boundary conditions at the fixed surface $Z = 0$, being:

$$U(0) = U'(0) = 0. \quad (23)$$

The two remaining boundary conditions are given by Eqs.(15, 16). The first rewrites:

$$U'' + g^4 k^2 U = 0 \quad \text{at } Z = H \quad (24)$$

which fixes

$$B = \frac{2 \cosh(g^2 H k) - \cosh(g^2 H k q) - q^2 \cosh(g^2 H k q)}{-2q \sinh(g^2 H k) + \sinh(g^2 H k q) + q^2 \sinh(g^2 H k q)}. \quad (25)$$

The second boundary condition on the free surface reduces to:

$$U''' + (2 + q^2)k^2 g^4 U' = 0 \quad \text{at } Z = H \quad (26)$$

and leads to the following dispersion relation:

$$4(q + q^3) - q(5 + 2q^2 + q^4) \cosh(g^2 H k) \cosh(g^2 H k q) + (1 + 6q^2 + q^4) \sinh(g^2 H k) \sinh(g^2 H k q) = 0. \quad (27)$$

Eq.(27) yields the classical Biot results, with a surface instability for $kH \rightarrow \infty$ for the real root q^{th} of $(q^3 + q^2 + 3q - 1) = 0$, being:

$$q^{th} = \frac{1}{3} \left[-1 - \frac{4 \cdot 2^{2/3}}{(13 + 3\sqrt{33})^{1/3}} + \left(2 \left(13 + 3\sqrt{33} \right) \right)^{1/3} \right] \cong 0.2955 \quad (28)$$

which in case of static wave, i.e. $V = 0$, imposes a critical $g^{th} = (q^{th})^{-1/2} \cong 1.8392$ or, equivalently, a critical compression $\lambda_x^{th} = 1/g^{th} = 0.5436$, **which is the limit calculated by Biot.**

Let us now consider a small but finite $\ell \ll H$ in order to analyze the second-gradient correction to the Biot instability. In particular, it is easy to show in the static case that $\Delta \bar{\Delta} f = 0$ is still a solution, so that U can be expressed by Eq. (21). Accordingly, the boundary condition in Eq.(24) is unaltered, while Eq.(26) is replaced by:

$$g^2 k^2 (1 + 2g^4 - g^2 k^2 \ell^2) U' - (g^2 + k^2 \ell^2) U''' = 0 \quad (29)$$

Accordingly, the corrected dispersion relation in this case is given by:

$$\begin{aligned} & 4g^6 (1 + g^4) - (1 + 2g^4 + 5g^8) k^2 \ell^2 \\ & + g^2 [(-1 - 2g^4 - 5g^8 + 4g^2 (1 + g^4) k^2 \ell^2) \cosh(Hk) \cosh(g^2 Hk) \\ & + (g^2 + 6g^6 + g^{10} - 2(1 + g^4)^2 k^2 \ell^2) \sinh(Hk) \sinh(g^2 Hk)] = 0 \end{aligned} \quad (30)$$

which allows us to eliminate the singularity at the instability threshold (i.e. the fact that the wavelength would be zero in absence of second gradient effects) thanks to the competition between the microstructural length ℓ and the layer's thickness H . In particular, the solution of the dispersion relation in Eq.(30) is depicted by the solid lines in Figure 1, showing both the growth threshold g^{th} and the critical dimensionless wavenumber $k^{th} H$ as a function of the ratio ℓ/H .

Finally, it is important to underline that such an approximation results from Eqs.(17,18) in non-vanishing hyperstresses at the free surface of order $(\ell\epsilon)^2$, and therefore can be considered valid only for $\ell/H \ll 1$. In such a range, it is possible to derive by series

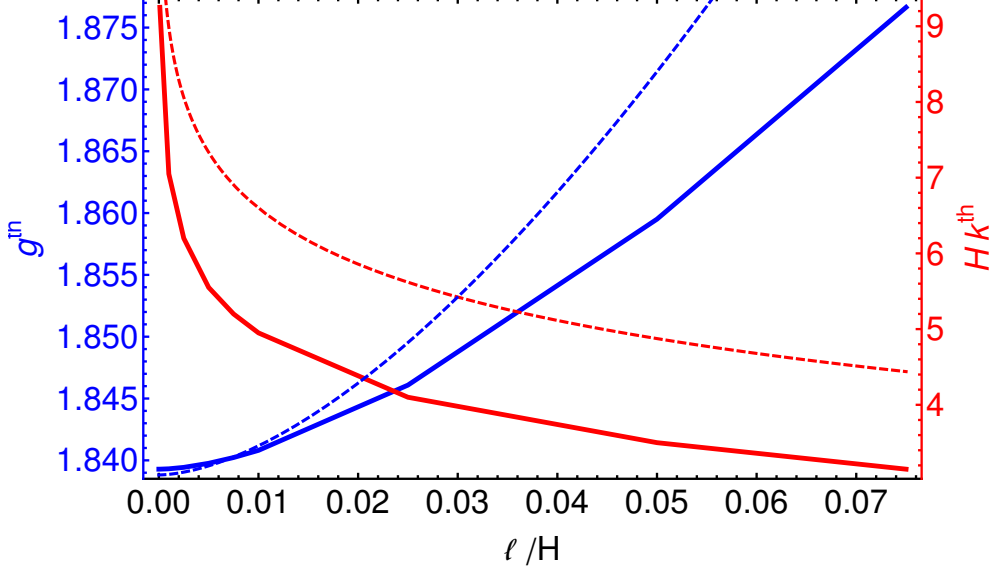


Figure 1: Solution of the dispersion relation in Eq.(30), showing both the growth threshold g^{th} (blue curves) and the critical dimensionless wavenumber $k^{th}H$ (red curves) versus the characteristic microstructural ratio ℓ/H . The solid lines is the numerical solution obtained by using the Newton method, while the dotted lines are the analytical approximation in Eq.(31).

development an analytical approximation of the instability thresholds, which reads:

$$k^{th} = \frac{1}{2H} \log \left[\frac{4.985}{0.129(\ell/H)^2} \right]; \quad g^{th} = 1.839 + 0.129 * (\ell/H)^2 \log \left[\frac{4.985}{0.129(\ell/H)^2} \right]^2 \quad (31)$$

and which is depicted as dotted lines in Figure 1

3.3 Neutrally-stable mode and near-critical waves

The neutrally-stable mode for the growing layer is defined by setting $V = 0$ in Eq.(22), whose solution is $\lambda_x = q^{1/2} = 1/g^{th} \simeq 0.5436$. If we consider a small increment of the growth from the linear stability threshold, such that $(g - g^{th})/g^{th} \ll 1$, from Eq.(22) we

derive that the velocity of the near-critical wave will scale as:

$$V \propto I \frac{\sqrt{4\mu}}{g^{th} \sqrt{\rho}} \sqrt{\frac{g - g^{th}}{g^{th}}} \quad (32)$$

Accordingly, considering a characteristic length $L_c = 1/k^{th}$ and a characteristic time $t_c = g^{th} L_c \sqrt{\rho/4\mu}$, the near-critical state can be characterized by the following series development:

$$\epsilon = \sqrt{\frac{g - g^{th}}{g^{th}}}; \quad \tau = \epsilon \frac{t}{t_c}; \quad \Psi(x, Z, t) = g^2 x Z + \sum_{i=1}^{\infty} \epsilon^i \Psi^{(i)}(x, Z, t) \quad (33)$$

where τ is the characteristic slow time of the near-critical wave and $\Psi^{(i)}$ are the series development of the perturbation on the stream function. In particular, considering the solution of the linear stability analysis in Eq.(21) and the scaling in Eq.(33), the first-order perturbation of the stream function reads:

$$\Psi^{(1)}(x, Z, t) = A(\tau) U(Z) \exp[Ikx] + c.c. \quad (34)$$

where $U(Z)$ is the linear (real) solution in Eq.(21), and $A(\tau)$ is the wave amplitude which shall be fixed by nonlinear effects. In the following, a weakly nonlinear analysis will be performed using the scaling arguments and the series developments in Eqs.(33).

4 Weakly nonlinear stability analysis

In this section, a weakly nonlinear analysis for the growing layer will be performed. In the following the second gradient terms will be neglected in the derivation of the second order solution: although even a very small ℓ/H is found to fix the wavelength at the linear order, it is easy to show that it has little influence at higher orders.

In the following, we calculate both the second- and the third-order solutions of the Euler-Lagrange equations in Eq.(12) for the stream function.

4.1 Second-order solution for the stream function

The second-order perturbation of the stream function must contain quadratic nonlinear interactions with the linear solution, so it can be expressed as follows:

$$\Psi^{(2)} = U^{(2)}(Z)|A(\tau)|^2 + (I V^{(2)}(Z)A^2(\tau) \exp[2Ikx] + c.c.) \quad (35)$$

Accordingly, from Eq.(12) the bulk Euler Lagrange equation at the second order in ϵ imposes at order $|A(\tau)|^2$:

$$U^{(2)''''} = 0 \quad (36)$$

whilst the surface conditions from Eq.(15,16) read:

$$U^{(2)'''}(H) = U^{(2)''}(H) = 0 \quad (37)$$

which together with the boundary condition at the fixed boundary, i.e. $U^{(2)}(0) = U^{(2)'}(0) = 0$, impose that $U^{(2)}(Z) = 0$. This is a main difference with the results obtained with the method over incremental deformations superposed on a finite strain [17], where such a term does not vanish.

By equating the coefficient of $A^2(\tau) \exp[2Ikx]$, the bulk Euler Lagrange equation from Eq.(12) at the second order in ϵ rewrites:

$$\begin{aligned} & 16g^6k^4V^{(2)} + 3g^4k^3U(-2k^2U' + U''') \\ & + k \left\{ -4g^2(1 + g^4)kV^{(2)''} - 5U''U''' + U' [3(2 + g^4)k^2U'' - U''''] \right\} + g^2V^{(2)''''} = 0 \end{aligned} \quad (38)$$

At the same orders, the surface conditions at $Z = H$ from Eq.(15,16) read:

$$4g^6k^2V^{(2)} + kU' (g^4k^2U - U'') + g^2V^{(2)''} = 0 \quad (39)$$

$$g^4k^5U^2 + (-3 + 2g^4)k^3U'^2 - 3g^4k^3UU'' + 2k [2g^2(1 + 2g^4)kV^{(2)'} + U''^2] + kU'U''' - g^2V^{(2)''''} = 0 \quad (40)$$

while the boundary condition at the fixed boundary impose $V^{(2)}(0) = V^{(2)'}(0) = 0$.

The general solution of Eq.(38) can be written as:

$$V^{(2)} = e^{-2g^2kZ}c_1 + e^{-2kZ}c_2 + e^{2g^2kZ}c_3 + e^{2kZ}c_4 + a_1UU' - a_2U'U'' \quad (41)$$

where c_1, \dots, c_4 are constant parameters which are fixed by the boundary conditions, and

$$a_1 = \frac{g^2(-41 + 9g^4)k}{9 - 82g^4 + 9g^8}; \quad a_2 = \frac{15(-1 + g^4)}{g^2(9 - 82g^4 + 9g^8)k} \quad (42)$$

The boundary conditions at the fixed layer impose:

$$c_1 + c_2 + c_3 + c_4 = 0 \quad (43)$$

$$2k(-c_2 + g^2(-c_1 + c_3) + c_4) - \frac{15(-1 + g^4)U''(0)^2}{g^2(9 - 82g^4 + 9g^8)k} = 0 \quad (44)$$

whilst from Eqs.(39, 40) the stress-free conditions at $Z = H$ rewrite:

$$2(9 - 82g^4 + 9g^8) \left[2e^{2Hk}g^4c_1 + e^{2g^2Hk}(1 + g^4)c_2 + 2e^{2(1+2g^2)Hk}g^4c_3 + e^{2(2+g^2)Hk}(1 + g^4)c_4 \right] + e^{2(1+g^2)Hk}g^2(-49 - 184g^4 + 105g^8)kU(H)U'(H) = 0 \quad (45)$$

$$\begin{aligned} & -e^{2Hk}(1 + g^4)c_1 - 2e^{2g^2Hk}g^2c_2 + e^{2(1+2g^2)Hk}(1 + g^4)c_3 + 2e^{2(2+g^2)Hk}g^2c_4 \\ & -e^{2(1+g^2)Hk} \left[3g^4(-3 + 5g^4 - 37g^8 + 35g^{12})k^2U(H)^2 + 2(9 - 115g^4 + 173g^8 - 3g^{12})U'(H)^2 \right] = 0 \end{aligned} \quad (46)$$

The analytical expression of c_1, \dots, c_4 is very cumbersome and it is not written out for the sake of simplicity.

4.2 Third-order solution for the stream function

The third-order perturbation of the stream function must contain both linear and cubic nonlinear interactions with the linear solution, as follows:

$$\Psi^{(3)} = (W^{(3)}(Z)A^2(\tau)\bar{A}(\tau) \exp[Ikx] + V^{(3)}(Z)A^3(\tau) \exp[3Ikx] + c.c.) \quad (47)$$

At order $A^2(\tau)\bar{A}\exp[Ikx]$, the bulk Euler Lagrange equation from Eq.(12) at the third order in ϵ takes the form:

$$\begin{aligned}
& g^4 W^{(3)''''} - g^4 k(1 + g^4) W^{(3)''} + g^8 k^4 W^{(3)} + \\
& g^4 k^4 U^2 (3U'''' - k^2 U'') \\
& + g^4 k^3 U \left(9kU''^2 - k^3 U'^2 + 12kU'''U' + 6g^2 k^2 V^{(2)'} - 3g^2 V^{(2)''''} \right) \\
& + k \left(4g^2 U''' V^{(2)''} + g^2 V^{(2)''''} U'' + kU''^3 \right) \\
& + 2g^2 V^{(2)'} [U'''' - 3(g^4 - 1)k^2 U''] + kU'^2 [U'''' + 6(g^4 - 3)k^2 U''] \\
& + U' [4kU'''U'' - g^2 V^{(2)''''} - 6(g^4 - 1)g^2 k^2 V^{(2)''}] = 0
\end{aligned} \tag{48}$$

At the same orders, the surface conditions at $Z = H$ from Eq.(15,16) read:

$$\begin{aligned}
& k \left(3g^4 k^3 U^2 U'' + 2g^2 U'' V^{(2)'} + 4g^6 k^2 V^{(2)} U' - g^2 U' V^{(2)''} + kU'^2 U'' \right. \\
& \left. - 2g^6 k^2 U V^{(2)'} + g^8 k W^{(3)} \right) + g^4 W^{(3)''} = 0
\end{aligned} \tag{49}$$

$$\begin{aligned}
& g^4 k^4 U^2 (k^2 U' - 3U''') + g^4 k^3 U \left(-6kU'U'' - 4g^2 k^2 V^{(2)} + 3g^2 V^{(2)''} \right) \\
& + k \left\{ 6k^3 U'^3 + g^2 \left[g^2 (2g^4 + 1) kW^{(3)'} - 2 \left(U''' V^{(2)'} + U'' V^{(2)''} \right) \right] \right. \\
& \left. - kU'''U'^2 + U' \left[k \left(2g^2 (2g^4 - 3) kV^{(2)'} - U''^2 \right) + g^2 V^{(2)''''} \right] \right\} - g^4 W^{(3)''''} = 0
\end{aligned} \tag{50}$$

while the boundary condition at the fixed boundary impose $W^{(3)}(0) = W^{(3)'}(0) = 0$.

The general form of the solution of Eqs.(48-50) has a very complex analytical expression, which is not given in the following for the sake of brevity.

Using the same procedure, the solution for $V^{(3)}$ could be derived at the order $\exp[3Ikx]$.

Since it can be checked that it will give rise to higher order terms in the energy contribution, it will not be considered in the following.

Having calculated the second- and third-order solutions for the problem, we will derive in the following section the amplitude equations for the elastic instability.

5 Derivation of the amplitude equations for the perfect system

The second-order solutions of the incremental problem contain nonlinear quadratic terms of interaction with the linear mode. A further series development of the solution would introduce resonant terms in the third-order displacement fields, because of the cubic nonlinear interactions of the linear modes. Therefore, a number of *amplitude* or *Ginzburg-Landau* equations should be introduced in order to avoid the occurrence of secular terms in the solution of the incremental equations.

The full derivation of the third-order elastic solution of this problem would be very hard to get and it is not considered in the following. Alternatively, the variational approach allows us to define a more elegant method for deriving the amplitude equations based on energy considerations.

In fact, applying the Hamilton's principle with respect to the Lagrangian density \mathcal{L} in Eq.(8) and imposing invariance with respect to simple translations allow us to define the following conservation principle [1]:

$$\frac{d}{dt} \iiint_V \left(\frac{d\mathcal{L}}{d\mathbf{v}} \mathbf{v} - \mathcal{L} \right) dx dz = \frac{d}{dt} \iiint_{\bar{V}} (K + W) \Psi_{,xZ} dx dZ = 0 \quad (51)$$

stating that the total mechanical energy $E = (K + W)$ must be conserved for the continuum body.

Let us consider a grown state with $g = g^{th}(1 + \epsilon^2)$: making the series expansion of E in terms of ϵ , the terms at the order ϵ^3 vanish for the imposed periodicity on the x direction. Moreover, also the term in ϵ^2 is zero, because of the condition of linear marginal stability for a static elastic solution, which imposes $\delta\mathcal{L} = \delta W = 0$ (in fact, in incremental terms it would be the volume integral of $1/2 \cdot S_{ij} u_{i,j}$, the power between the incremental stress \mathbf{S} and the gradient of the incremental displacement \mathbf{u} , which is zero thanks to boundary conditions and incompressibility - see section 4.4 of [17] for further details on this).

Therefore the total mechanical energy at the lowest order in ϵ is given by:

$$\begin{aligned}
E = & \frac{(-1+g^{th2})^2 \pi \mu H}{k} + \int_{Z=0}^H \frac{1}{g^{th8} k} \pi \mu \left\{ g^{th8} (1 + 3g^{th4}) + 8g^{th4} \left| \frac{\partial A(\tau)}{\partial \tau} \right|^2 \left(g^{th4} k^2 U^2 + U'^2 \right) \right. \\
& - 2g^{th4} |A(\tau)|^2 \left[g^{th4} k^4 U^2 + \left(5 + 3g^{th4} \right) k^2 U'^2 + 2g^{th4} k^2 U U'' + 5U''^2 \right] \\
& + 2|A(\tau)|^4 \left[16g^{th8} k^4 V^{(2)2} + 2g^{th8} k^2 W^{(3)} U'' + 6g^{th8} k^2 U' W^{(3)'} + 12g^{th8} k^2 V^{(2)'}^2 + 6g^{th6} k^3 U'^2 V^{(2)'} \right. \\
& + 2g^{th4} k^2 U' W^{(3)'} + 4g^{th4} k^2 V^{(2)'}^2 + 2g^{th4} U'' W^{(3)''} + g^{th4} V^{(2)''2} - 6g^{th2} k^3 U'^2 V^{(2)'} \\
& + 2g^{th2} k U''^2 V^{(2)'} - 2g^{th2} k U' U'' V^{(2)''} \\
& + 2g^{th6} k^2 U \left(g^{th2} k^2 W^{(3)} + g^{th2} W^{(3)''} - 2k U'' V^{(2)'} + k U' V^{(2)''} \right) \\
& + 8g^{th6} k^2 V^{(2)} \left(g^{th2} V^{(2)''} - k^3 U U' + k U' U'' \right) \\
& \left. + g^{th4} k^4 U^2 \left(2g^{th2} k V^{(2)'} + k^2 U'^2 + 3U''^2 \right) + 3k^4 U'^4 + k^2 U'^2 U''^2 \right\} \epsilon^4 dZ + O(\epsilon^4)
\end{aligned} \tag{52}$$

Accordingly, recalling that the mechanical energy must be a conserved quantity, the Ginzburg-Landau equation arise by imposing $dE/d\tau = 0$, being:

$$\frac{d^2 A(\tau)}{d\tau^2} = r A(\tau) + \nu |A(\tau)|^2 A(\tau) \tag{53}$$

where the conjugate equation is also imposed. From Eq.(52), the linear stability coefficient reads $r = -\alpha_1/\beta$ and the nonlinear term is $\nu = -2\alpha_2/\beta$, with:

$$\begin{aligned}
\beta = & \int_{Z=0}^H 8g^{th4} \left(g^{th4} k^2 U^2 + U'^2 \right) dZ; \\
\alpha_1 = & - \int_{Z=0}^H 2g^{th4} \left[g^{th4} k^4 U^2 + \left(5 + 3g^{th4} \right) k^2 U'^2 + 2g^{th4} k^2 U U'' + 5U''^2 \right] dZ; \\
\alpha_2 = & \int_{Z=0}^H 2 \left[16g^{th8} k^4 V^{(2)2} + 2g^{th8} k^2 W^{(3)} U'' + 6g^{th8} k^2 U' W^{(3)'} + 12g^{th8} k^2 V^{(2)'}^2 \right. \\
& + 2g^{th4} k^2 U' W^{(3)'} + 4g^{th4} k^2 V^{(2)'}^2 + 2g^{th4} U'' W^{(3)''} + g^{th4} V^{(2)''2} - 6g^{th2} k^3 U'^2 V^{(2)'} \\
& + 6g^{th6} k^3 U'^2 V^{(2)'} + 2g^{th2} k U''^2 V^{(2)'} - 2g^{th2} k U' U'' V^{(2)''} \\
& + 2g^{th6} k^2 U \left(g^{th2} k^2 W^{(3)} + g^{th2} W^{(3)''} - 2k U'' V^{(2)'} + k U' V^{(2)''} \right) \\
& + 8g^{th6} k^2 V^{(2)} \left(g^{th2} V^{(2)''} - k^3 U U' + k U' U'' \right) \\
& \left. + g^{th4} k^4 U^2 \left(2g^{th2} k V^{(2)'} + k^2 U'^2 + 3U''^2 \right) + 3k^4 U'^4 + k^2 U'^2 U''^2 \right] dZ
\end{aligned} \tag{54}$$

The amplitude equations in Eq.(53) correspond to a pitchfork bifurcation, which will be supercritical (resp. subcritical) if $r/\nu < 0$ (resp. $r/\nu > 0$). As expected, the term r only depends on the linear incremental solution [23]. Such a specific normal form

of the bifurcation is imposed by the symmetry of the system equations. In fact, the conservation of the mechanical energy imposes $E(t) = E(t + t_0)$ requiring invariance after a transformation $A(\tau) \rightarrow A(\tau)e^{t_0/t_c}$, and it is easy to check that an inversion invariance $A(\tau) \rightarrow -A(\tau)$ (or equivalently $U \rightarrow -U$, $V^{(2)} \rightarrow V^{(2)}$) also applies in Eq.(52), thus forcing the normal form in Eq.(53).

Finally, substituting the expressions of $U(Z)$, $V^{(2)}(Z)$, $W^{(3)}(Z)$ given by Eqs.(21,41) into Eq.(53), the coefficient of the Ginzburg-Landau equations are derived numerically, as collected in Table 1 for different values of the microstructural length ℓ .

ℓ/H	0.01	0.025	0.05	0.075
r	1.00339	1.01523	1.04366	1.07971
ν	$3.9763 \cdot 10^6$	229403	29675.7	9982.5

Table 1: Coefficients of the Ginzburg-Landau equations for different values of the ratio between the microstructural length ℓ and the layer's thickness H

In particular, for small values of ℓ/H the pitchfork bifurcation is always subcritical, meaning that this weakly nonlinear analysis cannot identify a stable branch of the bifurcated solution. Thus, nonlinear effects of higher order will drive the transition towards a given pattern. Even if the fully nonlinear morphology cannot be completely captured, the resulting deformation in this weakly nonlinear analysis show the tendency to create an elastic singularity, i.e. the formation of a crease [24], as the amplitude $A(\tau)$ **increases as the layer grows beyond the instability threshold g^{th}** , as depicted in Figure 2.

In fact, these results show that incipient cusp-like structures may occur at very low amplitude values, having the characteristic morphology observed experimentally by Tanaka and coworkers [26] in their early experiments on swelling gels under spatial confinement. Moreover, the results in Table 1 also highlight that increasing the ratio between the microstructural length ℓ and the layer's thickness H has a stabilizing effect on the wrinkling instability, possibly turning the bifurcation supercritical beyond a critical value of ℓ/H .

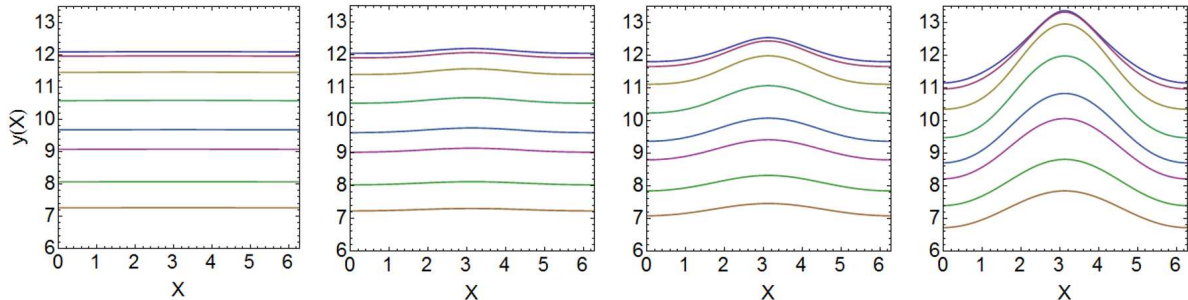


Figure 2: Numerical solution of the elastic displacement in $z(X)$ as a function of the initial axial position X , shown at different initial planes Z for $\ell/H = 0.05$, setting $\epsilon A(\tau) = (0.0001, 0.001, 0.002, 0.003)$ from left to right.

This stabilizing effect is somewhat in accordance to the findings for a soft compressed substrate coated with a thin elastic layer [6, 8], where stable wrinkles emerge if the film stiffness is sufficiently bigger than the substrate's.

However, we can only claim here that the amplitude equations prove that the instability is subcritical in this regime, since the weakly nonlinear solution for the stream function was found using the assumption that $\ell/H \ll 1$. The subcriticality suggests that the elastic instability can be highly sensitive to surface imperfections in the layer's free surface, as found for the compressed half space [7]. Thus, this effect will be further investigated in the following section.

6 Amplitude equations for a layer with surface imperfections

Let us now perform a sensitivity analysis of the growth instability in the presence of surface imperfections. In particular, the top surface is taken to have a sinusoidal profile (i.e. a modal imperfection) such that the vertical extension of the layer is given by:

$$Z \in \left[0, H + \xi^2 \frac{2\pi}{k_{imp}} (e^{Ik_{imp}x} + e^{-Ik_{imp}x}) \right] \quad (55)$$

where k_{imp} is the wavenumber of the imperfection and $\xi^2 \ll 1$ is a small dimensionless parameter, so that the the gradient of the initial surface elevation is of order $O(\xi^2)$. Accordingly the $O(1)$ problem represents the perfect layer under homogeneous compression. Considering that the sinusoidal mode forces the solution at the order ξ^2 , we can assume at the order ξ an incremental solution in the form of a Fourier summation, being:

$$\Psi(x, Z, t) = g^2 Z x + \xi \cdot \Psi^{(1)}; \quad \text{with } \Psi^{(1)}(x, Z, t) = \sum_{n \neq 0} A_n(\tau) U_n(Z, n) \exp[Ik_{imp} n x] \quad (56)$$

where n is an integer, the series goes from $-\infty$ to $+\infty$, $A_n(\tau)$ is amplitude of the mode n , and $U_n(Z)$ is the solution of the incremental problem in Eq.(21), taken for $k = (|n| k_{imp})$. In particular, from the linear stability analysis in the perfect system it is easy to show that $A_{-n}(\tau) = \bar{A}_n(\tau)$ and $U_{-n}(Z) = U_n(Z)$. The choice of this scaling is dictated by the observation that, even though setting a small $\ell/H \ll 1$ allows us to fix the first unstable mode k^{th} , the linear stability analysis also highlights that small wavelength modes have become unstable soon beyond the critical growth value g^{th} . Accordingly, the surface imperfection can have a cubic resonance with each of the **infinitely many** linearly unstable modes, as shown in the following. Let us look for an asymptotic solution in the form:

$$g = g^{th}(1 - \xi \cdot \alpha); \quad \tau = \sqrt{\xi \alpha} \frac{t}{t_c}; \quad \Psi(x, Z, t) = g^2 x Z + \sum_{i=1}^{\infty} \xi^i \Psi^{(i)}(x, Z, t) \quad (57)$$

where α is of order $O(1)$ and defines the increment of the growth. It is possible to show that the solution at order ξ^2 cannot be found unless a solvability condition is imposed on the incremental (leading order) solution $\Psi^{(1)}$. In fact, making the series expansion of the mechanical energy E on ξ , one finds the lowest order non-vanishing term at order ϵ^3 .

Recalling that:

$$\int_{Z=0}^{H+\xi^2 \eta(x)} (K+W) \Psi_{,xZ} dx dZ = \int_{Z=0}^H (K+W) \Psi_{,xZ} dx dZ + \xi^2 \eta(x) |(K+W) \Psi_{,xZ}|_{Z=H} \quad (58)$$

with $\eta(x) = \frac{2\pi}{k_{imp}}(e^{Ik_{imp}x} + e^{-Ik_{imp}x})$, the total mechanical energy at the lowest order in ξ is given by:

$$E = \frac{(-1+g^{th2})^2 \pi \mu H}{k_{imp}} + \xi^3 \left[\sum_{n \neq 0} \left(+\nu_n \left| \frac{\partial A_n(\tau)}{\partial \tau} \right|^2 + \beta_1 |A_n(\tau)|^2 \right. \right. \\ \left. \left. + \sum_{\substack{m \neq 0 \\ m \neq n}} Q(m, n) A_m(\tau) A_{n-m}(\tau) \bar{A}_n(\tau) \right) + B + \beta_2 (A_1(\tau) - \bar{A}_1(\tau)) \right] + O(\xi^3) \quad (59)$$

with:

$$B = -\frac{\mu \pi H [(g^{th})^4 - 1]}{k_{imp}} \quad (60)$$

$$\nu_n = \frac{4\mu\pi}{(g^{th})^4} \int_{Z=0}^H \left[(U'_n)^2 + n^2 k_{imp}^2 (g^{th})^4 U_n^2 \right] dZ \quad (61)$$

$$\beta_1 = \frac{\pi \mu \alpha}{(g^{th})^4} \int_{Z=0}^H \left[n^2 k_{imp}^2 (5 + 3g^{th4}) (U'_n)^2 + 5(U''_n)^2 + 2n^2 k_{imp}^2 g^{th4} U''_n U_n - n^4 k_{imp}^4 g^{th4} U_n^2 \right] dZ \quad (62)$$

$$\beta_2 = -I \frac{\mu \pi^2 [3(g^{th})^4 - 2(g^{th})^2 - 1]}{k_{imp} (g^{th})^2} k_{imp} U'_1(H) \quad (63)$$

$$Q(m, n) = I \frac{\mu \pi}{(g^{th})^6} \int_{Z=0}^H \left[(g^{th4} - 1) k_{imp}^3 m n (n - m) U'_m U'_n U'_{m-n} \right. \\ \left. + n k_{imp} U'_n U''_m U''_{m-n} + 2g^{th4} k_{imp}^3 m^2 (n - m) U''_n U_m U'_{m-n} + g^{th4} k_{imp} m U''_n U'_m U''_{m-n} \right] dZ \quad (64)$$

The series development in Eq. (59) is invariant after transformations $n \rightarrow -n$ and $m \rightarrow -m$, which correspond to $A_n \rightarrow -\bar{A}_n$, $A_m \rightarrow -\bar{A}_m$, $Q(n, m) = Q(-n, -m)$. Such a symmetry will force the expression of the normal form of the bifurcation, i.e. the Ginzburg-Landau equations for the system with imperfections. In fact, the conservation of the mechanical energy imposes that $\frac{dE}{d\tau} = \sum_{\alpha} \left[\frac{\partial E}{\partial A_{\alpha}} A'_{\alpha}(\tau) + \frac{\partial E}{\partial A'_{\alpha}} A'_{\alpha}(\tau) \right] = 0$, leading to an infinite number of amplitude equation in the following form:

$$\frac{d^2 A_{\alpha}(\tau)}{d\tau^2} = -\frac{\beta_1}{\nu_n} \cdot A_{\alpha}(\tau) - \frac{\beta_2}{2\nu_n} (\delta_{1\alpha} - \delta_{-1\alpha}) - \frac{1}{2\nu_n} \sum_{m \neq \alpha} K(\alpha, m) A_m A_{\alpha-m} \quad (65)$$

with $K(\alpha, m) = (Q(-m, -\alpha) + Q(\alpha, m) + Q(-m, \alpha - m))$. In particular, recalling that $A_{-\alpha} = \bar{A}_\alpha$, the amplitude equations admit a solution $A_\alpha = IB_\alpha$ with B_α being a real number, such that $B_{-\alpha} = -B_\alpha$. This is driven by the mentioned symmetry restriction $A_{-\alpha} = \bar{A}_\alpha$ in the expression of the mechanical energy. In this case, the amplitude equations in Eq.(65) can be expressed in the following real form:

$$\frac{\partial^2 B_\alpha(\tau)}{\partial \tau^2} = -\frac{\beta_1}{\nu_n} \cdot B_\alpha(\tau) + I \frac{\beta_2}{2\nu_n} (\delta_{1\alpha} - \delta_{-1\alpha}) - \frac{I}{2\nu_n} \sum_{m \neq \alpha} K(\alpha, m) B_m B_{\alpha-m} \quad (66)$$

where δ indicates the Kronecker delta. Eq.(66) has the same expression found in [15] using the virtual work method.

Let us now look for the static solution of Eq.(66) truncating the system at the second order (i.e. setting $N=2$ and assuming $B_M = 0$ with $M > N$). By straightforward calculations, we find that $B_2 = (IB_1^2 K(2, 1))/(2\beta_1)$, whilst B_1 is given by solving:

$$I\beta_2 + 2B_1\beta_1 - \frac{B_1^3 [K(1, -1) + K(1, 2)] K(2, 1)}{2\beta_1} = 0 \quad (67)$$

In particular, B_1 will be given by the only real solution of Eq. (67) which tends to zero as α goes to infinity, respecting the fact that the wrinkle amplitude is of order of the surface imperfection in the absence of growth. Developing the terms in Eq.(67), it is found that such a root is real only if $\alpha > \alpha_{cr} = 0.867$, possibly indicating the presence of a turning point in the solution which is typical of elastic structures with high sensitivity to imperfections. The resulting second-order amplitudes are depicted in Figure 3.

We finally solved numerically Eq.(66) setting a higher truncation number N : we used an iterative technique increasing the truncation number M in unit steps, i.e. $M = (N + 1)$, setting as initial guess the solution from the previous step N together with $B = N + 1 = 0$ until a the convergence criterion is reached (specifically on the increment of the sum of the Fourier amplitudes). In particular, it is found that the inclusion of higher modes has a negligible effect on the amplitude of the wrinkling, although it has a significant steepening effect on the surface profile, possibly indicating the incipient formation of an elastic singularity in the fully nonlinear regime.

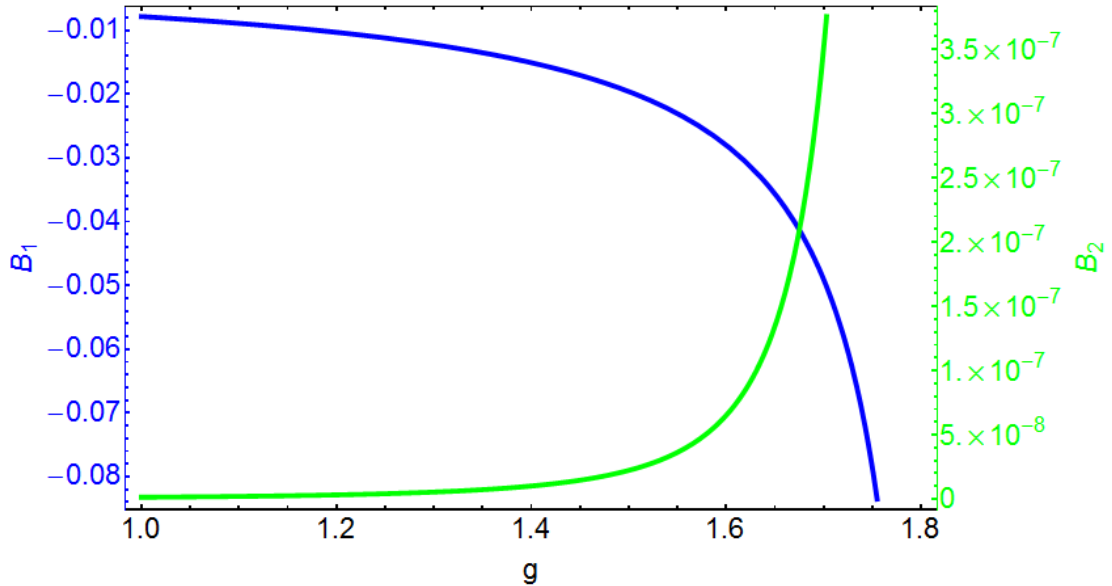


Figure 3: Amplitudes B_1 and B_2 of the weakly nonlinear solution truncated at $N = 2$ versus the layer's growth g . The initial surface imperfection was set at $\xi = 0.005$

7 Conclusion

In this paper we have used a semi-analytical approach to investigate the post-buckling behavior in a swelling hyperelastic layer of thickness H which is fixed to a rigid substrate. We considered the presence of strain gradient effects through the introduction of a microstructural length ℓ in its constitutive behavior in Eq.(2), so that the Biot instability can be studied approaching the limit $\ell/H \rightarrow 0$.

We introduced a nonlinear stream function for enforcing exactly the incompressibility constraint, developing a variational formulation to perform the stability analysis of the basic homogeneous solution. At the linear order, the bulk incremental equations of motion are given in a novel form by Eq.(13). The resulting dispersion relation is given by Eq.(27), demonstrating that even a small strain gradient effect, i.e. $\ell/H \ll 1$, can regularize the wavelength singularity of the Biot problem, allowing the system to select a critical dimensionless wavenumber while giving a small correction to the instability threshold.

We later performed a weakly nonlinear analysis by applying a multiple-scale expansion to

this neutrally stable mode. We derived a scaling for such a near-critical state in Eq.(33), later calculating both the second- and the third-order solutions of the incremental equations of motion for the stream function. By applying the global conservation of the mechanical energy, we were able to derive the Ginzburg-Landau equation for the critical mode in Eq. (53), which identifies a pitchfork bifurcation. In particular, it is found that the buckling is subcritical for small values of the ratio ℓ/H , so that pattern formation will be driven by nonlinear effects of higher order, which are out of reach of a weakly nonlinear analysis. Nonetheless, the derived solution seems to indicate the tendency towards the fully nonlinear formation of an elastic singularity at finite time, i.e. a crease.

Since the subcritical bifurcations are known to be very sensitive to defects, we finally considered the effect of the initial presence of a sinusoidal imperfection on the free surface of the layer. In this case, we have written the incremental solution for the stream function as a Fourier series, and we have derived a scaling in Eq.(57) so that the surface imperfection can have a cubic resonance with the linear modes. Using the proposed energetic approach, we derived an infinite series of amplitude equations for the linear modes in Eq.(65), which have been solved numerically for a truncated system. The solutions indicate the presence of a turning point close to the critical threshold for the perfect system, also finding that the inclusion of higher modes has a steepening effect on the surface profile. **This behavior for the imperfect system is qualitatively similar to the one previously investigated using the Koiter's method [7]. In fact, the subcriticality of wrinkling makes the layer highly sensitive to surface imperfections, also lowering the critical compressive strain as the square of their initial amplitude.**

In conclusion we have proposed a novel semi-analytical approach providing further insight on the post-buckling behavior of the Biot instability. Since we found that the pitchfork bifurcation of the neutrally-stable mode is subcritical in the limit $\ell/H \ll 1$, the weakly nonlinear results cannot describe the fully nonlinear dynamics of the problem, although suggesting the incipient formation of a static-shock or a crease. Nonetheless the proposed

theoretical results can be used for improving the existing finite element methods for simulations, providing the exact analytical results for guiding the numerical solutions in the post-buckling regime.

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