Dispersion of elastic waves in a layer interacting with a Winkler foundation

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Dispersion of plane harmonic waves in an elastic layer interacting with one or two-sided Winkler foundation is analysed. The long-wave low frequency polynomial approximations of the full transcendental dispersion relations are derived for a relatively soft foundation. The validity of the conventional engineering formulation of a Kirchhoff plate resting on an elastic foundation is investigated. It is shown that this formulation has to be refined near the cut-off frequency of bending waves. The associated near cut-off expansion is obtained for both cases. A simple explicit formula demonstrating veering of bending and extensional waves is presented for a one-sided foundation.

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I. INTRODUCTION

Dispersion of elastic waves in a free layer was investigated in great detail beginning with the seminal contribution of Lamb (Lamb, 1917). The symmetry of the problem significantly simplifies analysis enabling separation of all the quantities of interest into even and odd parts with respect to the mid-plane. Violation of the symmetry, due to interaction with the environment along one of the faces, leads to more sophisticated dispersion relations which cannot be generally reduced to simpler ones for symmetric and antisymmetric waves. As an example, we mention (Kaplunov and Markushevich, 1993) investigating vibration of elastic layer resting on an acoustic half-space. A layer supported by a Winkler foundation is another important asymmetric setup.

Bending of elastic structures on a Winkler foundation is usually treated in the framework of approximate engineering theories beginning with static analysis of an Euler-Bernoulli beam, see (Frỳba, 2013) and references therein. Associated dynamic formulations are always obtained just by incorporating an extra inertial term. However, such elementary trick does not take into consideration the peculiarities of the bending wave propagation in the vicinity of the cut-off frequency arising from the effect of the foundation. Formally, it corresponds to the absence of the leading order term in the Taylor near cut-off expansion expressed in terms of squared wavenumber. This phenomenon is also characteristic of formal dynamic generalization of the original static semi-membrane equations for a thin cylindrical shell, see (Kaplunov *et al.*, 2016a; Kaplunov and Nobili, 2017b; Strozzi *et al.*, 2014).

The current paper is aimed at revising the traditional 1D bending problem for a Kirchhoff plate on a Winkler foundation starting from the exact solution of the plane strain problem in linear elasticity with the main focus on a near cut-off behaviour. The fundamental extensional mode with a zero cut-off frequency arising in the 2D formulation, is also tackled in the paper.

The studied Winkler foundation may be treated as the leading order approximation of the related problem for an elastic laminate subject to appropriate boundary conditions along the faces, see (Aghalovyan, 2015). It still finds numerous applications including modelling of transit and edge bending waves, (Brun *et al.*, 2013), (Kaplunov *et al.*, 2016b), (Kaplunov *et al.*, 2014), see also (Kaplunov and Nobili, 2017a) using refined Pasternak model. Among the publications on the subject we also mention (Elishakoff *et al.*, 2018; Li *et al.*, 2009; Ponnusamy and Selvamani, 2012).

The paper is organized as follows. The dynamic equations in plane elasticity written in terms of wave potentials are presented in Section II. Along with the boundary conditions corresponding to the traditional 'one-sided' Winkler foundation, a symmetric 'two-sided' foundation is also considered. Various dispersion relations are derived in Section III. As might be expected, the dispersion relations for symmetric and antisymmetric waves may be separated from each other only for a two-sided foundation. The asymptotic expansions corresponding to the bending cut-off frequencies are obtained for a relatively soft Winkler foundation. Sections IV and V deal with long wave low frequency polynomial approximations of the transcendental dispersion relations derived in Section III for two and one-sided foundations, respectively. The leading order polynomial approximations of the antisymmetric dispersion relation in Section IV appears to contain a few extra terms in comparison with the conventional dynamic model of a plate on a Winkler foundation. The associated near cut-off expansion is also presented. The leading order approximation of the symmetric dispersion relation appears to be valid not only for a soft foundation and corresponds to the longitudinal extensional wave in a plate subject to a transverse compression. Asymptotic considerations in the last section lead to similar conclusions as in previous section. However, the related leading order polynomial dispersion relation and near cut-off asymptotic expansion take a more sophisticated form due to interaction between extensional and bending waves. Veering of the associated dispersion curves is specially emphasized.

II. STATEMENT OF THE PROBLEM

Consider the plane strain problem for an elastic plate of thickness $2h$ either embedded in a Winkler elastic foundation with stiffness θ, Fig. 1(a), or supported by the latter along the lower face with traction free upper face, Fig. 1(b). We adapt the Lamé decomposition

$$\boldsymbol{u} = \operatorname{grad}\phi + \operatorname{curl}\boldsymbol{\psi}, \tag{1}$$

of the two-dimensional displacement vector $\boldsymbol{u} = (u_1, u_2)$ through the scalar potential $\phi(x, y, t)$ and the only nonzero component of the vector potential $\psi_z(x, y, t)$ both satisfying the wave equation in two dimensions ($-\infty < x < \infty$, $-h \le y \le h$)

$$\Delta_2\phi - \frac{1}{c_1^2}\phi_{,tt} = 0, \tag{2}$$

and

$$\Delta_2\psi_z - \frac{1}{c_2^2}\psi_{z,tt} = 0, \tag{3}$$

with longitudinal and transverse wave speeds given by

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_2 = \sqrt{\frac{\mu}{\rho}},$$

where λ and μ are material constants, ρ is mass density, and $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2D Laplace operator, with comma in suffix denoting partial differentiation.

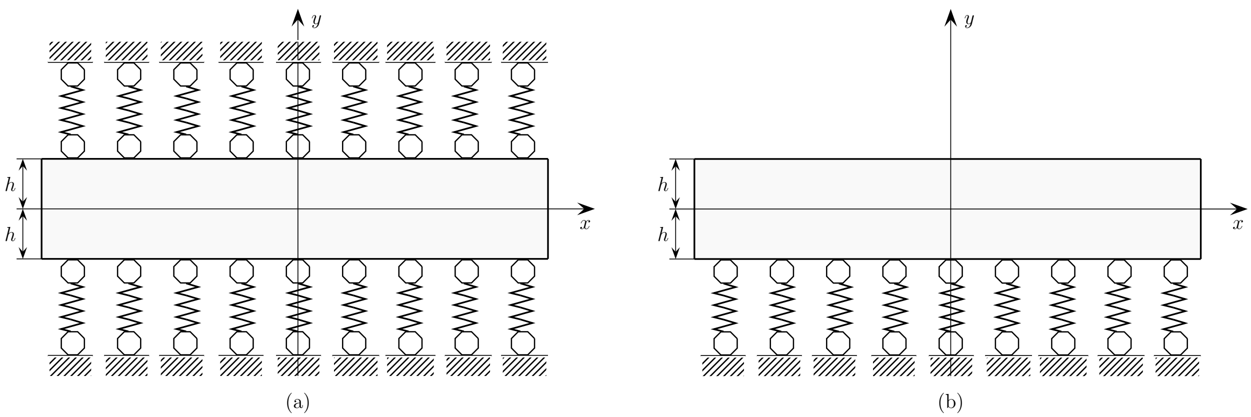

FIG. 1. Thin elastic layer embedded in a Winkler elastic foundation

The boundary conditions along the faces of the one-sided and two-sided foundations, see Fig. 1, respectively, take the forms,

$$\sigma_{22} = \mp\theta u_2 \quad \text{and} \quad \sigma_{12} = 0, \quad \text{at } y = \pm h. \tag{4}$$

and

$$\sigma_{22} = 0 \quad \text{at} \quad y = h; \quad \sigma_{22} = \theta u_2 \quad \text{at} \quad y = -h; \quad \sigma_{12} = 0 \quad \text{at} \quad y = \pm h. \tag{5}$$

The components of the displacement vector and stresses entering the boundary conditions (4) and (5) are expressed through the potentials as, e.g. see (Achenbach, 2012),

$$u_1 = \phi_{,x} + \psi_{z,y}, \quad u_2 = \phi_{,y} - \psi_{z,x}. \tag{6}$$

and

$$\sigma_{22} = 2\mu \left(\phi_{,yy} - \psi_{z,xy}\right) + \lambda \Delta_2 \phi,$$
$$\sigma_{12} = \mu \left(2\phi_{,xy} + \psi_{z,yy} - \psi_{z,xx}\right). \tag{7}$$

The main goal of the paper is to justify and refine the well-known low frequency engineering model for bending vibration of a plate supported by a Winkler foundation. For the configuration in Fig 1 (b), it is given by, e.g. see (Achenbach, 2012),

$$\frac{2Eh^3}{3(1-\nu^2)}\frac{\partial^4 w}{\partial x^4} + 2h\rho\frac{\partial^2 w}{\partial t^2} + \theta w = 0 \tag{8}$$

where $w \approx u_2(x, 0)$, $E = 2(1 + \nu)\mu$ is Young's modulus, and ν is Poisson's ratio.

Originally the Winkler model was adapted for static equilibrium of a beam (Winkler, 1870). The inertial term was formally added at the latest stage as a D'Alembert force. Let us show that such approach does not seem to be always justified. On substituting the travelling wave solution $w = \exp i(kx - \omega t)$ where ω is frequency and k is wave number, into equation (8), we get

$$\omega^2 - \frac{\theta}{2h\rho} = \frac{Eh^2k^4}{3\rho(1-\nu^2)}. \tag{9}$$

This relation may be treated as a Taylor expansion of a more general dispersion relation near the cut-off frequency $\omega = (\theta/2h\rho)^{1/2}$. Then, it is not very clear, why this expansion does not contain a term with k^2, but only with k^4 ! It is interesting that such issue also arises in the dynamic version of the semi-membrane theory for thin cylindrical shell, see (Kaplunov and Nobili, 2017b).

In addition, we note that apart from aforementioned bending vibrations, the formulated 2D plane strain problem also support extensional vibrations usually neglected in the considerations on the subject.

III. DISPERSION RELATIONS

We shall look for the solutions in the form of travelling waves

$$\phi(x,y,t)=\Phi(y)\exp i(kx-\omega t) \quad \text{and} \quad \psi_z(x,y,t)=\Psi_z(y)\exp i(kx-\omega t).$$

Then, we have from (2) and (3)

$$\Phi(\eta)=e_1\cos(\alpha\eta)+o_1\sin(\alpha\eta)\,, \tag{10}$$

$$\Psi_z(\eta)=o_2\cos(\beta\eta)+e_2\sin(\beta\eta)\,, \tag{11}$$

where $\eta=y/h$ and

$$\alpha^2=\chi^2\Omega^2-K^2, \quad \beta^2=\Omega^2-K^2, \tag{12}$$

with $\chi=c_2/c_1=\sqrt{(1-2\nu)/2(1-\nu)}$ and

$$K=kh \quad \text{and} \quad \Omega=\frac{\omega h}{c_2} \tag{13}$$

Consider first a two-sided foundation, see Fig. 1a. For symmetric waves when $o_i=0$, $i=1,2$ in (10) and (11), we substitute the latter into boundary conditions (4) taking into account the formulas (7). As a result, we arrive at the dispersion relation (cf. (Graff, 1975, Eq.(8.1.54)))

$$\frac{4\alpha\beta K^2}{\left(\beta^2-K^2\right)^2}+\frac{\tan\beta}{\tan\alpha}+G\frac{\alpha\Omega^2}{\left(\beta^2-K^2\right)^2}\tan\beta=0, \tag{14}$$

where $G=\theta h/\mu$ is the dimensionless stiffness of the Winkler foundation. Numerical illustration of the dispersion curve (14) is shown in Fig. 2. Here and henceforth, if not stated explicitly we take $G=0.01$ and $\nu=0.25$.

Similarly, for antisymmetric waves ($e_i=0$, $i=1,2$ in equations (10) and (11)) we have (cf. (Graff, 1975, Eq.(8.1.59)))

$$\frac{\left(\beta^2-K^2\right)^2}{4\alpha\beta K^2}+\frac{\tan\beta}{\tan\alpha}-G\frac{\Omega^2}{4\beta K^2}\cot\alpha=0. \tag{15}$$

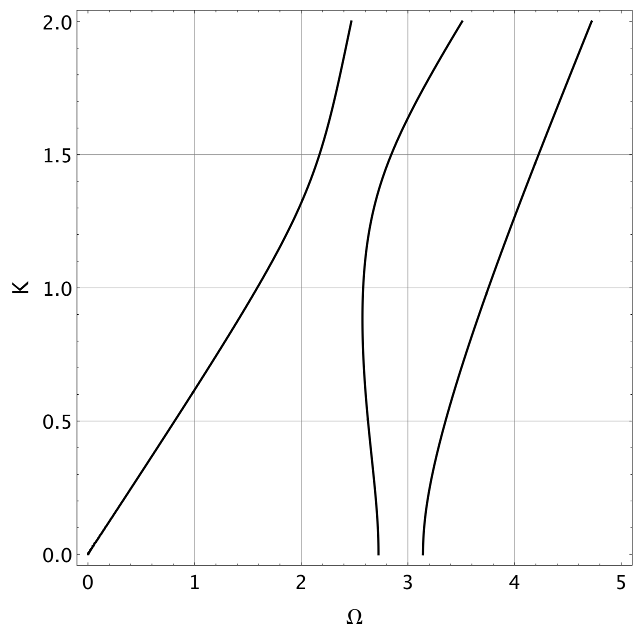

FIG. 2. Symmetric waves governed by the dispersion relation (14) for $G = 0.01$ and $\nu = 0.25$.

This dispersion relation is shown in Fig. 3. For a one-sided foundation, see Fig. 1(b), symmetric and antisymmetric waves are coupled with each other, resulting in four non-zero constants e_i and o_i, $i = 1, 2$ in (10) and (11). Then, we insert the latter into the 'nonsymmetric' boundary conditions (5) using formulae (7). The derived dispersion relation takes a more sophisticated form than above, and can be written as

$$\begin{aligned} &\frac{G\Omega^2}{16}\left(\gamma^4 \cos 2\alpha \frac{\sin 2\beta}{\beta} + K^2\alpha^2 \cos 2\beta \frac{\sin 2\alpha}{\alpha}\right) - \\ &- \left[\left(\frac{\gamma^8}{4\alpha\beta} + \frac{K^4\alpha\beta}{4}\right) \sin 2\beta \sin 2\alpha - \frac{\gamma^4 K^2}{2} \cos 2\beta \cos 2\alpha + \frac{\gamma^4 K^2}{2}\right] = 0 \end{aligned} \tag{16}$$

where $\gamma^2 = K^2 - \Omega^2/2$.

We remark that it is similar in a sense to the dispersion relation for an elastic plate lying on an acoustic half-space, e.g. (Kaplunov and Markushevich, 1993).

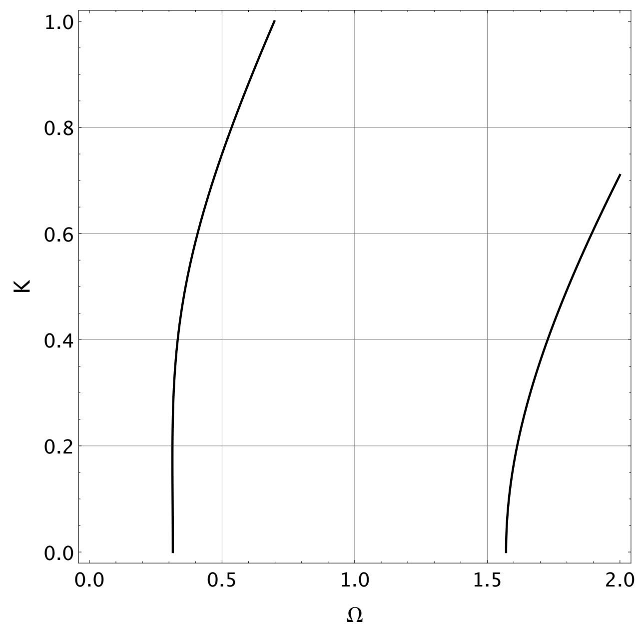

FIG. 3. Antisymmetric waves governed by the dispersion relation (15) for $G = 0.01$ and $\nu = 0.25$.

The cut-off frequencies ($K = 0$) of (16) affected by the Winkler foundation are given by

$$\tan 2\chi\Omega - \frac{G\chi}{\Omega} = 0 \tag{17}$$

For a relatively soft foundation ($G \ll 1$) the lowest cut-off frequency $\Omega = \Omega_*$ is expanded as

$$\Omega_* = \sqrt{\frac{G}{2}}\left(1 - \frac{G\chi^2}{3} + \cdots\right). \tag{18}$$

As might be expected, the leading order term in (18) coincides with the value of the cut-off frequency (9) originated from the approximate model (8).

Dispersion relation (16) possesses also zero cut-off frequency $\Omega = 0$, similarly to the dispersion relation (14), related to symmetric waves in a plate supported by a two-sided foundation. In the latter case, we obtain, at $G \ll 1$, from the dispersion equation for antisymmetric waves (15) the counterparts of the formulae (17) and (18). These take the

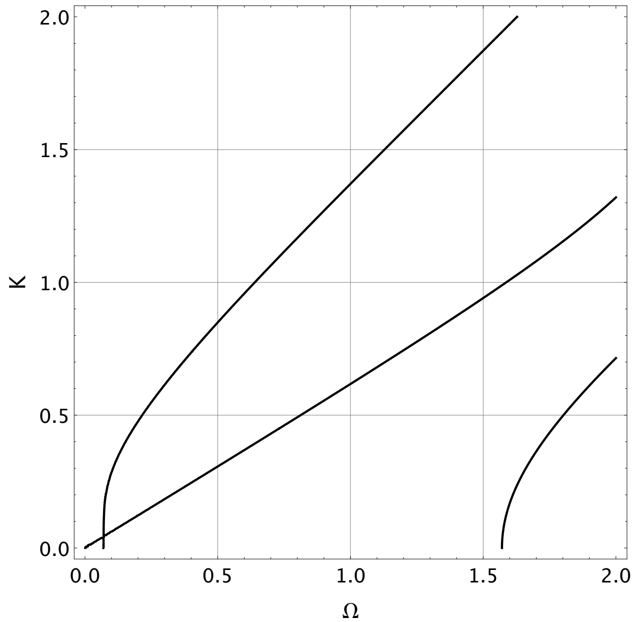

FIG. 4. Harmonic waves governed by the dispersion relation (16) for $G = 0.01$ and $\nu = 0.25$.

form

$$\tan(\chi\Omega) - \frac{G\chi}{\Omega} = 0. \tag{19}$$

and

$$\Omega_* = \sqrt{G}\left(1 - \frac{G\chi^2}{6} + \dots\right). \tag{20}$$

IV. LONG-WAVE LOW FREQUENCY APPROXIMATION FOR TWO-SIDED FOUNDATION

A. Antisymmetric Motion

At long wave, low frequency limit

$$\Omega \ll 1 \quad \text{and} \quad K \ll 1 \tag{21}$$

transcendental dispersion relation (15) under the assumption of a relatively weak Winkler foundation $G \ll 1$ has a polynomial expansion. At leading order it takes the form

$$K^4 - \frac{(1+\nu)}{2}K^2\Omega^2 - \frac{3(1-\nu)}{2}\Omega^2 - \frac{(1-2\nu)}{4}\Omega^4 + \frac{3(1-\nu)}{2}G = 0 \tag{22}$$

At $K = 0$, it supports two-term asymptotic formula (20) for cut-off frequency Ω_*.

Over the vicinity of the cut-off frequency $\delta = \Omega^2 - G \ll G$, the polynomial equation (22) to within asymptotically small terms may be rewritten as

$$\delta = \frac{G^2(1-2\nu)}{6(1-\nu)} + \frac{2}{3(1-\nu)}K^4 - \frac{1+\nu}{3(1-\nu)}K^2G + \cdots . \tag{23}$$

Note that at $\delta \gg G^2$, when $K \gg G^{1/2}$, this expansion reduces to

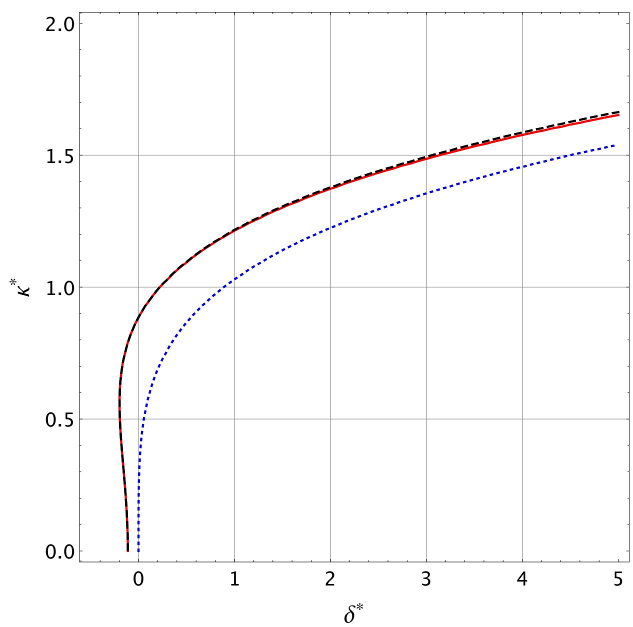

FIG. 5. Antisymmetric waves governed by (16) (black, solid), (23) (red, dashed), and (24) (blue dotted) for $G = 0.01$ and $\nu = 0.25$ in the scaled variables $\delta^* = \delta/G^2$ and $\kappa^* = K/\sqrt{G}$.

$$K^4 - \frac{3(1-\nu)}{2}\Omega^2 + \frac{3(1-\nu)}{2}G = 0, \tag{24}$$

corresponding to the conventional engineering model of a Kirchhoff plate resting on a Winkler foundation, see Section II.

It is clear that the inertial term in equation (24) may be neglected at $\Omega \ll G^{1/2}$ resulting in quasi-static behaviour

$$K^4 + \frac{3(1-\nu)}{2} G = 0. \tag{25}$$

On the other hand, at $\Omega \gg G^{1/2}$ we arrive at the dispersion relation for a free plate given by

$$K^4 - \frac{3(1-\nu)}{2} \Omega^2 = 0. \tag{26}$$

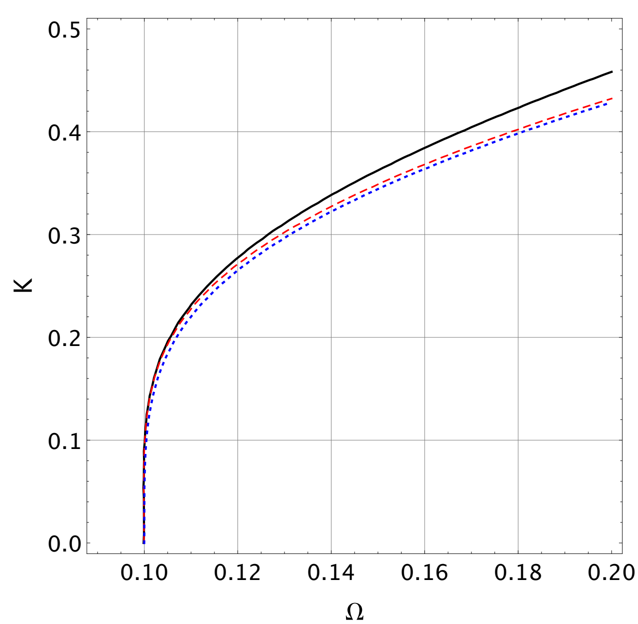

FIG. 6. Antisymmetric waves governed by (16) (black, solid), (23) (red, dashed), and (24) (blue dotted) for $G = 0.01$ and $\nu = 0.25$.

We also remark that near cut-off approximation (23) corresponds to the equation of motion

$$\frac{2Eh^3}{3(1-\nu^2)}\left(\frac{\partial^2}{\partial x^2}+\frac{\theta(1+\nu)^2}{Eh^3}\right)\frac{\partial^2 w}{\partial x^2}+ \\ +2h\rho\frac{\partial^2 w}{\partial t^2}+\theta\left(1-\frac{\theta(1-2\nu)(1+\nu)}{3Eh(1-\nu)}\right)w=0 \tag{27}$$

containing extra terms in comparison with the traditional formulation (8).

B. Symmetric Motion

Now, we have at leading order from transcendental dispersion relation (14) over the domain (21)

$$K^2=\frac{(2-2\nu)+G(1-2\nu)}{2\left(2+G(1-\nu)\right)}\Omega^2 \tag{28}$$

not making yet additional assumptions on the dimensionless parameter G. At $G \ll 1$, the latter can be expanded as

$$K^2\left(1+\frac{\nu^2 G}{2(1-\nu)}+\cdots\right)=\frac{1-\nu}{2}\Omega^2 \tag{29}$$

corresponding to the approximate model of plate transverse compression, (Kaplunov *et al.*, 1998), see also Fig 1, governed by the one dimensional equation

$$\frac{2Eh}{1-\nu^2}\frac{\partial^2 u_1}{\partial x^2}-2\rho h\frac{\partial^2 u_1}{\partial t^2}=-\frac{2h\nu}{1-\nu}\frac{\partial Q}{\partial x} \tag{30}$$

where $Q=\mp\theta u_2$ and $u_2=\mp\dfrac{\nu h}{1-\nu}\dfrac{\partial u_1}{\partial x}$.

As it might be expected, at $G=0$, the dispersion relation (29) coincides with that in the classical plane stress theory given by

$$K^2=\frac{1-\nu}{2}\Omega^2 \tag{31}$$

At $G \gg 1$ we have from (28), at leading order,

$$K^2 = \frac{1-2\nu}{2-2\nu}\Omega^2, \tag{32}$$

which also may easily be deduced from the 2D plane strain limiting problem, see Section 2, with the following mixed boundary conditions along the faces

$$\sigma_{12} = 0, \quad u_2 = 0 \quad \text{at} \quad y = \pm h. \tag{33}$$

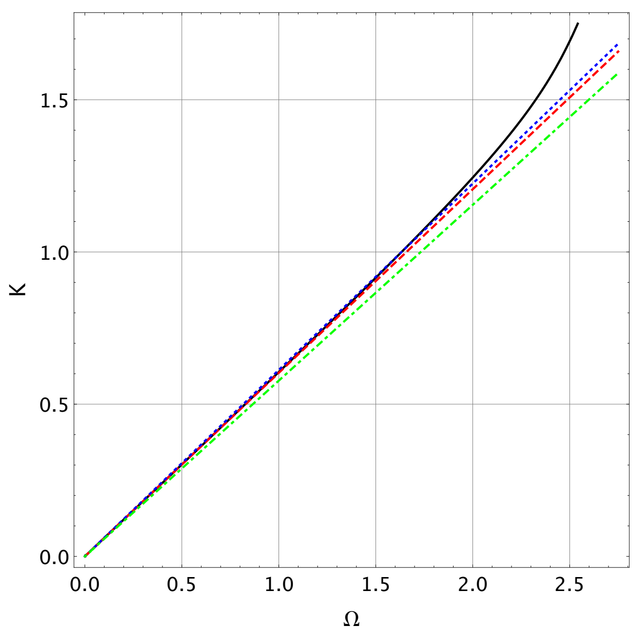

FIG. 7. Symmetric waves governed by the dispersion relations (16) (solid, black), (28) (red, dashed), (31) (blue dotted), and (32) (green, dot-dashed) for $G = 1$ and $\nu = 0.25$.

V. LONG-WAVE LOW FREQUENCY APPROXIMATION FOR ONE SIDED FOUNDATION

In this case, symmetric and antisymmetric plate motions are coupled with each other resulting in a more sophisticated polynomial shortened form of the original transcendental dispersion relation (16). We have at leading order, assuming (21), and also $G \ll 1$,

$$\begin{gathered}\frac{G\Omega^2}{32} - \frac{G}{16(1-\nu)}K^2 + \frac{1}{8}\left(\frac{1}{1-\nu} + \frac{G(7-12\nu+4\nu^2)}{6(1-\nu)^2}\right)K^2\Omega^2 - \\ - \frac{1}{16}\left(1 + \frac{G(5-8\nu)}{6(1-\nu)}\right)\Omega^4 - \frac{G}{12(1-\nu)}K^4 + \frac{3-2\nu}{12(1-\nu)^2}\Omega^2 K^4 - \\ - \frac{(11-16\nu+4\nu^2)}{48(1-\nu)^2}\Omega^4 K^2 - \frac{1}{12(1-\nu)^2}K^6 + \frac{3-4\nu}{48(1-\nu)}\Omega^6 = 0\end{gathered} \tag{34}$$

Let us first transform this equation into two identical forms

$$\begin{gathered}\frac{1}{8(1-\nu)}\left(\Omega^2 - \frac{G}{2}\right)\left(K^2 - \frac{1-\nu}{2}\Omega^2\right) = -\frac{G\left(7-12\nu+4\nu^2\right)}{48(1-\nu)^2}K^2\Omega^2 + \\ + \frac{G\left(5-8\nu\right)}{96(1-\nu)}\Omega^4 + \frac{G}{12(1-\nu)}K^4 - \frac{3-2\nu}{12(1-\nu)^2}\Omega^2 K^4 + \\ + \frac{4\nu^2-16\nu+11}{48(1-\nu)^2}\Omega^4 K^2 - \frac{3-4\nu}{48(1-\nu)}\Omega^6 + \frac{1}{12(1-\nu)^2}K^6,\end{gathered} \tag{35}$$

and

$$\begin{gathered}\frac{1}{12(1-\nu)^2}K^2\left(K^4 - \frac{3(1-\nu)}{2}(\Omega^2 - G/2)\right) = \\ = \frac{G}{32}\Omega^2 - \frac{1}{16}\left(1 + \frac{G\left(5-8\nu\right)}{6(1-\nu)}\right)\Omega^4 - \frac{G}{12(1-\nu)}K^4 + \\ + \frac{3-2\nu}{12(1-\nu)^2}\Omega^2 K^4 - \frac{11-16\nu+4\nu^2}{48(1-\nu)^2}\Omega^4 K^2 + \frac{3-4\nu}{48(1-\nu)}\Omega^6.\end{gathered} \tag{36}$$

in order to get a better idea of the studied extension and bending waves, respectively. Next, we obtain the two-term asymptotic expansions of the latter

$$K^2 = \frac{1-\nu}{2}\Omega^2\left(1 - \frac{G\nu^2}{3\delta(1-\nu)}\Omega^2 + \frac{\nu^2}{6\delta(1-\nu)}\Omega^4 + \cdots\right), \tag{37}$$

and

$$K^4 = \frac{3(1-\nu)\delta}{2}\left(1 - \frac{1}{4\delta^{3/2}}\sqrt{\frac{2(1-\nu)}{3}}\left(4\delta G - \left(G + \frac{4(3-2\nu)}{1-\nu}\delta\right)\Omega^2 + 2\Omega^4\right) + \cdots\right) \tag{38}$$

where, now, $\delta = \Omega^2 - G/2$. They clarify the physical meaning of long wave low frequency approximation (34), in particular, the leading order term in (38) corresponds to the dispersion relation for a beam resting on an elastic foundation, see (8).

The derived formulae (37) and (38) are valid outside the vicinity of the lowest bending cut-off frequency, see (18), namely at, $\delta \gg G^2$ and $\delta \gg G^{4/3}$, respectively. Instead, at $\delta \ll G$, we arrive at a near cut-off expansion of (34) given by

$$\begin{aligned} \delta\left(1 - \frac{4}{G(1-\nu)}K^2 + \frac{2\delta}{G}\right) &= -\frac{G^2(1-2\nu)}{6(1-\nu)} + \\ &+ \frac{8}{3G(1-\nu)^2}\left(\frac{3-8\nu+4\nu^2}{16}K^2 + \frac{1}{2G}K^4 - \frac{1}{G^2}K^6\right) \end{aligned} \tag{39}$$

It is worth noting that at $\delta \gg G^2$ it coincides at leading order with (37) and (38) provided that $K \sim G^{1/2}$ and $K \gg G^{1/2}$, respectively. We also remark that near the point $\Omega^2 = G/2$ and $K^2 = G(1-\nu)/4$ both expansions (37) and (39) at leading order may reduce to

$$\delta_\Omega\left(\frac{1-\nu}{2}\delta_\Omega - \delta_K\right) = \frac{\nu^2}{32}, \tag{40}$$

where

$$\delta_\Omega = \frac{\Omega^2 - G/2}{G^{3/2}} \quad \text{and} \quad \delta_K = \frac{K^2 - (1-\nu)G/4}{G^{3/2}}.$$

This explicit formula illustrates veering (Mace and Manconi, 2012) of the extensional and the bending dispersional curves first noted in Section 3.

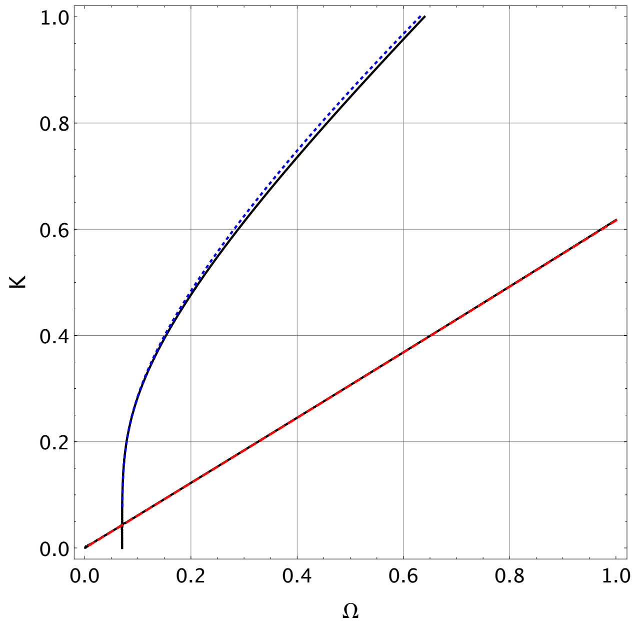

FIG. 8. Comparison of the dispersion curves corresponding to (34) (black solid), (37) (red, dashed), and (38) (blue dotted), for $G = 0.01$ and $\nu = 0.25$.

VI. CONCLUSIONS

The long wave low-frequency shortened polynomial forms of the ‘exact’ transcendental dispersion relations for an elastic layer interacting with two and one-sided Winkler foundations are analysed. It is shown that the traditional engineering approximation within the framework of the classical theory of plate bending is not uniformly valid. It fails near the cut-off frequencies characteristic of elastically supported structures. The near cut-off asymptotic expansions, see (23) and (39), are derived; in doing so, the expansion for a one-sided foundation takes a pretty sophisticated form, due to interaction of bending and extensional waves. At the same time, the simple explicit formula (40) visualising veering of two types of waves is obtained.

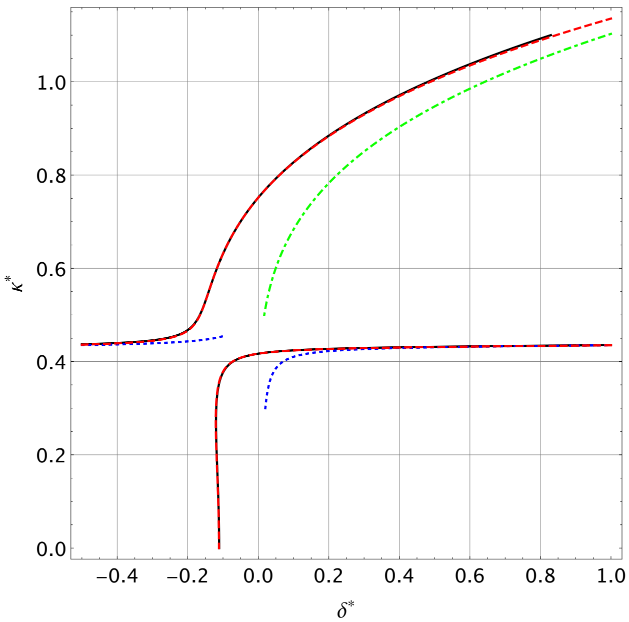

FIG. 9. Comparison of the dispersion curves corresponding to (34) (black solid), (39) (red, dashed), (37) (blue, dotted), and (38) (green, dotdashed), for $G = 0.01$ and $\nu = 0.25$ in the scaled variables $\delta^* = \delta/G^2$ and $\kappa^* = K/\sqrt{G}$.

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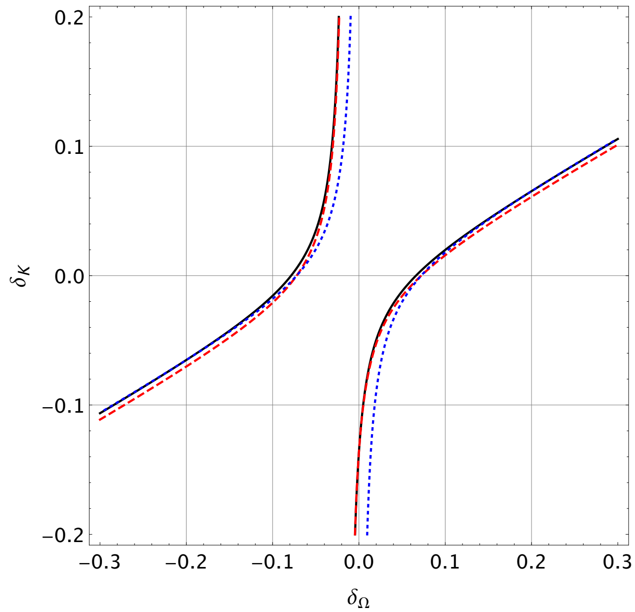

FIG. 10. Veering of the dispersion curves according to (34) (black, solid), (39) (red, dashed), and (40) (blue, dotted) for $G = 0.01$, $\nu = 0.25$ in the scaled variables δ_Ω and δ_K.

REFERENCES

Achenbach, J. (**2012**). *Wave propagation in elastic solids*, **16** (Elsevier).

Aghalovyan, L. (**2015**). *Asymptotic Theory Of Anisotropic Plates And Shells* (World Scientific).

Brun, M., Movchan, A. B., and Slepyan, L. I. (**2013**). "Transition wave in a supported heavy beam," Journal of the Mechanics and Physics of Solids **61**(10), 2067–2085.

Elishakoff, I., Tonzani, G. M., Zaza, N., and Marzani, A. (**2018**). "Contrasting three alternative versions of timoshenko–ehrenfest theory for beam on winkler elastic foundation–simply supported beam," ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik https://onlinelibrary.wiley.com/doi/abs/10.1002/zamm.201700019, doi: 10.1002/zamm.201700019.

Frỳba, L. (**2013**). *Vibration of solids and structures under moving loads*, **1** (Springer Science & Business Media).

Graff, K. F. (**1975**). *Wave motion in elastic solids* (Dover Publishing, Inc).

Kaplunov, J., Manevitch, L., and Smirnov, V. (**2016**a). "Vibrations of an elastic cylindrical shell near the lowest cut-off frequency," Proc. R. Soc. A **472**(2189), 20150753.

Kaplunov, J., and Markushevich, D. (**1993**). "Plane vibrations and radiation of an elastic layer lying on a liquid half-space," Wave Motion **17**(3), 199–211.

Kaplunov, J., and Nobili, A. (**2017**a). "The edge waves on a kirchhoff plate bilaterally supported by a two-parameter elastic foundation," Journal of Vibration and Control **23**(12), 2014–2022.

Kaplunov, J., and Nobili, A. (**2017**b). "A robust approach for analysing dispersion of elastic waves in an orthotropic cylindrical shell," Journal of Sound and Vibration **401**, 23–35.

Kaplunov, J., Prikazchikov, D., and Rogerson, G. (**2016**b). "Edge bending wave on a thin elastic plate resting on a winkler foundation," Proc. R. Soc. A **472**(2190), 20160178.

Kaplunov, J., Prikazchikov, D. A., Rogerson, G. A., and Lashab, M. I. (**2014**). "The edge wave on an elastically supported kirchhoff plate," The Journal of the Acoustical Society of America **136**(4), 1487–1490.

Kaplunov, J. D., Kossovitch, L. Y., and Nolde, E. (**1998**). *Dynamics of thin walled elastic bodies* (Academic Press).

Lamb, H. (**1917**). "On waves in an elastic plate," Proceedings of the Royal Society of London. Series A **93**(648), 114–128.

Li, Z.-D., Yang, T.-Q., and Luo, W.-B. (**2009**). "An improved model for bending of thin viscoelastic plate on elastic foundation," Natural Science **1**(02), 120.

Mace, B. R., and Manconi, E. (**2012**). "Wave motion and dispersion phenomena: Veering, locking and strong coupling effects," The Journal of the Acoustical Society of America **131**(2), 1015–1028.

Ponnusamy, P., and Selvamani, R. (**2012**). "Wave propagation in a generalized thermo elastic plate embedded in elastic medium," Interact. Multiscale Mech **5**(1), 13–26.

Strozzi, M., Manevitch, L. I., Pellicano, F., Smirnov, V. V., and Shepelev, D. S. (**2014**). "Low-frequency linear vibrations of single-walled carbon nanotubes: Analytical and numerical models," Journal of Sound and Vibration **333**(13), 2936–2957.

Winkler, E. (**1870**). *Vorträge über Eisenbahnbau: Gehalten am königl. böhmischen polytechnischen Landesinstitut in Prag*, **5** (H. Dominicus).