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Multi-parametric dynamic analysis of lightweight elastic laminates

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Abstract. Multi-parametric asymptotic analysis of dynamic phenomena in lightweight three-layered structures is performed. The presence of high contrast in densities of skin and core layers may lead to the small value of the lowest shear thickness resonance frequency, allowing a two-mode long wave low-frequency approximation of the exact dispersion relation. The range of uniform asymptotic validity of the two-mode approximation with respect to relative thickness and density are established. The theoretical predictions are illustrated numerically.

Introduction

Multi-layered structures with a strong vertical heterogeneity are of major importance for numerous high-tech domains in aerospace, automotive and civil engineering, see e.g. [1, 2] and references therein. Specific applications include, in particular, photovoltaic panels and laminated glass, see [3, 4, 5]. Another important example originating from automotive industry, is concerned with design of a new generation of environment-friendly lightweight vehicles, see [6, 7]. This inspires modelling of composite structural elements, containing lightweight components.

Multi-parametric asymptotic analysis of dispersion of elastic plates with high contrast in material and geometric parameters of the layers was studied in [8], see also a more recent contribution [9]. The key message of the cited papers is that due to contrast the lowest shear mode may be excited over the low frequency range, requiring substantial amendments to the established structural theories. In particular, instead of the classical Kirchhoff theory governing the fundamental vibration mode of a thin plate, a variety of two-mode approximations, strongly affected by the contrast parameters were discovered in [8]. The latter, in addition to the fundamental mode, also include the first low-frequency harmonic.

In this presentation we attempt to specify the approach in [8] for three-layered structures with a lightweight core component. In this case, we adapt a multi-parametric scheme assuming that the ratio of thickness of the skin and core component layers is expressed as a power of the main small parameter, defined as the relative core density.

The two-mode approximations of the full Rayleigh-Lamb-type dispersion relation, both uniformly and non-uniformly valid, are derived. The accuracy of the latter are tested by comparison with the exact solution. The obtained results address peculiarities of the vibration spectra characteristic of lightweight layered structures.

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Statement of the problem

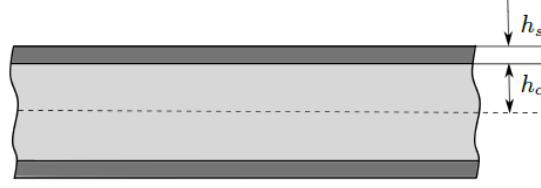


Figure 1. A lightweight sandwich plate.

Consider a sandwich symmetric plate with a light core, see Fig. 1, assuming that the ratio of densities of the core and skin layers

$$\rho = \frac{\rho_c}{\rho_s} \ll 1 \quad (1)$$

is a small parameter. In this case the inverse ratio of their thickness is expressed as

$$h = \frac{h_s}{h_c} \sim \rho^a, \quad (2)$$

where $0 \leq a < 1$. In addition, for the sake of simplicity we assume for the ratio of shear moduli

$$\mu = \frac{\mu_c}{\mu_s} \sim \rho. \quad (3)$$

Then, the lowest shear resonant frequency

$$\Omega_{sh} \approx \left(\frac{\rho}{h}\right)^{1/2} \sim \rho^{(1-a)/2} \ll 1, \quad (4)$$

belongs to the low-frequency range, see also an earlier contribution [10] for a three-component rod.

The consideration below deals with a multi-parametric analysis (in ρ and a) of the full dispersion relation of the sandwich structure of interest at the long-wave low-frequency limit, when

$$K = kh_c \ll 1, \Omega = \omega h_c \sqrt{\frac{\rho_c}{\mu_c}} \ll 1, \quad (5)$$

where ω and k are angular frequency and wave number, respectively.

Asymptotic analysis

First, expand the dispersion relation in [8] over range (5) under the assumptions on the problem parameters adapted in the previous section. As a result, we arrive at the polynomial expression

$$\gamma_1 \Omega^2 + \gamma_2 K^4 + \gamma_3 K^2 \Omega^2 + \gamma_4 K^6 + \gamma_5 \Omega^4 + \dots = 0, \quad (6)$$

where

$$\gamma_1 = -\mu(h\mu_0 + \mu),$$

$$\gamma_2 = -\frac{4}{3}\mu(h(h^2 + 3h + 3)(\kappa_s^2 - 1) + \mu(\kappa_c^2 - 1)),$$

$$\gamma_3 = \frac{2}{3}(\mu\mu_0 h^3(2\kappa_s^2 - 3) + 3h^2((\kappa_s^2 - 1)(\mu^2 + 2\mu_0) - \mu_0\mu\kappa_s^2) - 3\mu h(-2\kappa_s^2 + \mu_0 + 2) + \mu^2(2\kappa_c^2 - 3)),$$

$$\gamma_4 = \frac{4}{15}(\mu h(1 - \kappa_s^2)(10h\kappa_c^2(1 - \mu) + 5h^3\kappa_s^2 + 10h^2 + 10\mu h + 5 + h^4) + 5h^4\kappa_s^2(\kappa_s^2 - 2) + \mu^2(1 - \kappa_c^2) + 5h^4),$$

$$\gamma_5 = \frac{1}{6}(\mu^2(3\mu_0(\kappa_s^2 + 1)h^2 + \kappa_c^2 + 3) + \mu h \mu_0(h^2 \mu_0(\kappa_s^2 + 3) + 3\kappa_c^2 + 9) + 6h^2 \mu_0^2).$$

In the formulae above

$$\mu_0 = \frac{\mu}{\rho} \quad \text{and} \quad \kappa_q = \frac{c_{2q}}{c_{1q}},$$

with

$$c_{1q}^2 = \frac{\lambda_q + 2\mu_q}{\rho_q}, \quad c_{2q}^2 = \frac{\mu_q}{\rho_q}, \quad q = c, s.$$

The leading order asymptotic approximation of (6) corresponding to the fundamental mode, can be written as

$$\rho^{1-a} G_1 \Omega^2 + \rho^{1-a} G_2 K^4 + G_3 K^2 \Omega^2 + \frac{1}{3} \rho^{2a} G_2 K^6 = 0, \quad (7)$$

where $G_i \sim 1$, $i = 1, 2, 3$ are expressed as

$$G_1 = -h_0 \mu_0^2, \quad G_2 = -4h_0 \mu_0 (\kappa_s^2 - 1), \quad G_3 = 4h_0^2 \mu_0 (\kappa_s^2 - 1),$$

with $h_0 = h/\rho^a$.

The shortened forms of (7) are

$$G_1 \Omega^2 + G_2 K^4 = 0, \quad \text{at} \quad \Omega \ll \rho^{1-a}, \quad (0 \leq a < 1), \quad (8)$$

$$\begin{aligned} \rho^{1-a} G_2 K^2 + G_3 \Omega^2 = 0, \quad \text{at} \quad \rho^{1-a} \ll \Omega \ll \rho^{1-2a}, \quad \left(a \leq \frac{1}{3}\right), \\ \text{or} \quad \rho^{1-a} \ll \Omega \ll \rho^{\frac{1-a}{2}}, \quad \left(\frac{1}{3} < a < 1\right), \end{aligned} \quad (9)$$

and

$$G_3 \Omega^2 + \frac{1}{3} \rho^{2a} G_2 K^4 = 0, \quad \text{at} \quad \rho^{1-2a} \ll \Omega \ll \rho^a, \quad \left(a < \frac{1}{3}\right). \quad (10)$$

The leading order asymptotic behavior of the first harmonic with the cut-off frequency (4) takes the form

$$\rho^{1-a} G_1 + G_3 K^2 + G_5 \Omega^2 = 0, \quad \text{at} \quad \Omega - \Omega_{sh} \ll 1 \quad (0 \leq a < 1), \quad (11)$$

where for $G_5 \sim 1$ we have $G_5 = h_0^2 \mu_0^2$.

The derived formulae can be easily justified using the entries in Tables 1 and 2 below.

Table 1. Asymptotic behavior for the fundamental mode.

| Order of γ_i | Terms | Fundamental mode |
|---------------------|-------|------------------|
|---------------------|-------|------------------|

| | | | | |
|--|-------------------------|--|--|---|
| | | $\Omega^2 \sim K^4$ $\Omega \ll \rho^{1-a}, 0 \leq a < 1$ | $\Omega^2 \sim \rho^{1-a} K^2$ $\rho^{1-a} \ll \Omega \ll \rho^{1-2a}, a \leq \frac{1}{3}$ $\rho^{1-a} \ll \Omega \ll \rho^{\frac{1-a}{2}}, \frac{1}{3} < a < 1$ | $\Omega^2 \sim \rho^{2a} K^2$ $\rho^{1-2a} \ll \Omega \ll \rho^a, a < \frac{1}{3}$ |
| $\gamma_1 \sim \rho^{1-a}$ | $\gamma_1 \Omega^2$ | $\rho^{1-a} K^4$ | $\rho^{2-2a} K^2$ | $\rho^{1+a} K^4$ |
| $\gamma_2 \sim \rho^{1-a}$ | $\gamma_2 K^4$ | $\rho^{1-a} K^4$ | $\rho^{1-a} K^4$ | $\rho^{1-a} K^4$ |
| $\gamma_3 \sim 1$ | $\gamma_3 K^2 \Omega^2$ | K^6 | $\rho^{1-a} K^4$ | $\rho^{2a} K^4$ |
| $\gamma_4 \sim \rho^{2a} + \rho^{1-a}$ | $\gamma_4 K^6$ | $(\rho^{2a} + \rho^{1-a}) K^6$ | $(\rho^{2a} + \rho^{1-a}) K^6$ | $\rho^{2a} K^6$ |
| $\gamma_5 \sim 1$ | $\gamma_5 \Omega^4$ | K^8 | $\rho^{2-2a} K^4$ | $\rho^{4a} K^8$ |

Table 2. Asymptotic behavior for the lowest harmonic.

| Order of γ_i | Terms | Harmonic |
|--|-------------------------|---|
| | | $\Omega^2 - \Omega_{sh}^2 \sim K^2$ $\Omega - \Omega_{sh} \ll 1$ |
| $\gamma_1 \sim \rho^{1-a}$ | $\gamma_1 \Omega^2$ | $\rho^{1-a} (K^2 + \rho^{1-a})$ |
| $\gamma_2 \sim \rho^{1-a}$ | $\gamma_2 K^4$ | $\rho^{1-a} K^4$ |
| $\gamma_3 \sim 1$ | $\gamma_3 K^2 \Omega^2$ | $K^2 (K^2 + \rho^{1-a})$ |
| $\gamma_4 \sim \rho^{2a} + \rho^{1-a}$ | $\gamma_4 K^6$ | $(\rho^{2a} + \rho^{1-a}) K^6$ |
| $\gamma_5 \sim 1$ | $\gamma_5 \Omega^4$ | $(K^2 + \rho^{1-a})^2$ |

It is remarkable that as it follows from the Table 1 and also formulae (8)-(10), only at $0 \leq a \leq \frac{1}{3}$ the cut-off frequency belongs to the range of validity of the long-wave approximation (7) for the fundamental mode.

Thus, combining the expressions (7)-(11), we arrive at the uniformly valid two- mode approximation

$$\rho^{1-a} G_1 \Omega^2 + \rho^{1-a} G_2 K^4 + G_3 K^2 \Omega^2 + \frac{1}{3} \rho^{2a} G_2 K^6 + G_5 \Omega^4 = 0, \quad 0 \leq a \leq \frac{1}{3}. \quad (12)$$

At the same time, at $\frac{1}{3} < a < 1$, the cut-off frequency (4) is outside the range of validity of (7), see Table 1 and formulae (8)-(10). Consequently, the corresponding two-mode expansion

$$\rho^{1-a} G_1 \Omega^2 + \rho^{1-a} G_2 K^4 + G_3 K^2 \Omega^2 + G_5 \Omega^4 = 0, \quad \frac{1}{3} < a < 1, \quad (13)$$

is no more uniformly valid.

Numerical results

Let us illustrate the obtained results. Figs. 2 and 3 below are plotted for the values $\rho = 0.01, \mu = 0.01$, and $\kappa_s = 0.59$. The two-mode asymptotic approximation (12) along with the exact dispersion curves for $a = \frac{1}{9}, h = 0.6$ and $a = \frac{1}{3}, h = 0.23$ are shown in Figs. 2(a) and 2(b), respectively. It is

clear from Fig. 2(a) that the presented approximation is uniform, reaching its bound at $a = \frac{1}{3}$ in Fig. 2(b), see (12). Indeed, the vertical dotted lines, originating from the cutoff frequencies intersect the fundamental modes at $K < 1$.

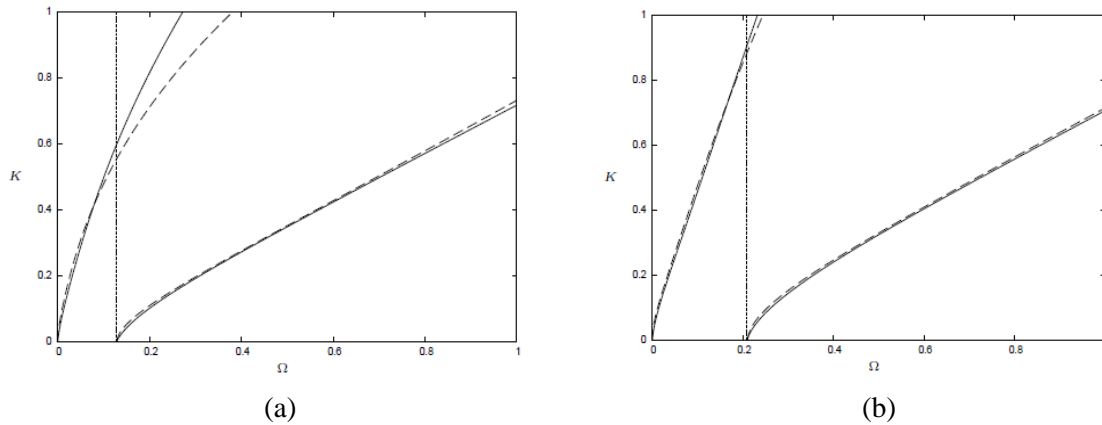


Figure 2. Comparison of two-mode approximations (12) (dashed lines) and the exact dispersion curves (solid lines) for $\rho = 0.01, \mu = 0.01, \nu_s = 0.59$ with (a) $a = \frac{1}{9}, h = 0.6$; (b) $a = \frac{1}{3}, h = 0.23$.

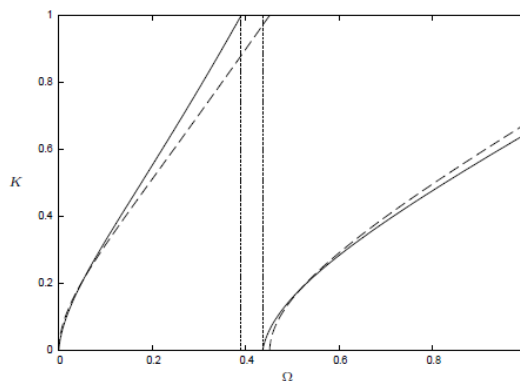


Figure 3. Comparison of two-mode composite approximation (13) (dashed lines) and the exact dispersion curves (solid lines) for $a = \frac{2}{3}, \rho = 0.01, \mu = 0.01, h = 0.05, \nu_s = 0.59$.

The composite expansion (13) versus the exact solution is shown in Fig. 3 for $a = \frac{2}{3}$ and $h = 0.05$. The non-uniform nature of (13), associated with a gap in frequency between the two vertical dotted lines can be clearly observed. Indeed, within this gap the long wave assumption ($K \ll 1$) is violated.

Concluding remarks

The developed approach is aimed at analysis and optimization of low-frequency vibration spectra of lightweight laminates. In particular, the consideration above revealed the range of parameters satisfying

$$\frac{h_s}{h_c} = \left(\frac{\rho_c}{\rho_s}\right)^a, 0 \leq a \leq \frac{1}{3}, \tag{14}$$

for which the asymptotic approximation (12) is uniform, whereas for the remaining region $\frac{1}{3} < a < 1$ the formula (13) provides only a composite approximation.

For plate bending, the shift of the first thickness shear resonance to the low frequency band results in its interaction with the fundamental mode. This makes the implementation of direct finite element computations more challenging, see e.g. [11], motivating a more elaborative theoretical insight.

The presented methodology is not restricted to the setup of a three-layered symmetric plate. It can be extended to other types of lightweight laminates. In addition, extra problem parameters, including a strong contrast in the wave speed for core and skin layers, may be taken into consideration. We also mention the generalizations to shell-type structures with thermo-viscoelastic core layers.

The derived two-mode expansions of the dispersion relation are an important preliminary step for establishing general models for lightweight multi-layered structures, including differential equations of motion along with properly justified boundary conditions.

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