Nonlinear Stability Analysis of a Pre-stressed Elastic Half-space

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ABSTRACT

In this paper the nonlinear evolution of near-neutral modes in a pre-stressed elastic half-space governed by an infinite system of evolution equations is discussed. The theory is illustrated for the case in which the pre-stress is a uniaxial compression and the perturbation consists initially of a single mode. It is shown that excitation of harmonics due to nonlinear interaction always leads to the formation of shocks, whether the elastic half-space is super-critically or sub-critically near-neutral and that when the half-space is super-critically near-neutral shocks always form before any significant growth in amplitude has taken place. In considering the static specialization of the evolution equations, two existing methods are assessed critically and shown to be flawed.

1. INTRODUCTION

It is well known that an elastic half-space can transmit (Rayleigh) surface waves. When the half-space is pre-stressed, the Rayleigh wave speed becomes a function of the pre-stress. At a pre-stress for which the Rayleigh wavespeed vanishes the half-space is said to be *neutrally stable*. When the wavespeed is small and real, the half-space is said to be *sub-critically near-neutral*, and when it is imaginary but has small magnitude the half-space is said to be *super-critically near-neutral*. The main objective of a linear stability analysis is to find the conditions, usually referred to as *bifurcation conditions*, under which neutral stability is attained. Linear stability of pre-stressed elastic half-spaces has been studied for various materials and forms of pre-stress by a number of authors (see [1–4] and the references therein). The purpose of the present paper is to examine nonlinear aspects of the theory.

The initial motivation is the need to clarify the question of the existence and determination of any post-buckling state (shape and amplitude) associated with a pre-stressed half-space. In a linear analysis, the bifurcation condition is independent of the forms and amplitudes of the instability modes, the amplitudes being undetermined. Determination of the amplitude in the present context, which involves non-dispersive modes, is a much harder problem than when dispersive modes are involved. In the latter case, exemplified by the buckling of pre-stressed elastic plates [5], it may be assumed that the leading-order solution takes the form of a single sinusoidal mode. Nonlinear interaction produces a mean field and the second harmonic at second order and, at third order, a resonance term, and the imposition of a solvability condition determines the amplitude of the leading-order solution. In the buckling of a pre-stressed half-space, the leading-order solution can no longer be assumed to consist of a single mode, as pointed out by Wu and Cao [2]. In the light of recent developments in nonlinear surface wave propagation in an *unstressed* half-space, it is clear that any nonlinear solution involving non-dispersive modes necessarily takes the form of an infinite Fourier sum (or a Fourier integral). A nonlinear stability analysis involving non-dispersive modes has been conducted recently for a pre-stressed, infinite, inextensible elastic body [6, 7].

In this paper, the dynamic problem of determining the evolution of near-neutral modes (i.e. modes with small wavespeed or growth rate) in a pre-stressed elastic half-space is considered first. An infinite sum is assumed for the nonlinear solution, and the evolution of the amplitudes of various Fourier modes, which depend on a slow time variable, is determined at second order, leading to an infinite set of coupled evolution equations. Determination of post-buckling states corresponds to the solution of the static form of the evolution equations. The derivation of these equations is similar to that for nonlinear surface waves in an unstressed half-space, and the mathematical problem of determining the post-buckling configurations is similar to that of determining nonlinear surface waves of permanent form in an unstressed half-space. The latter problem has been considered by two research groups with conflicting results. Parker and Talbot [8] claimed that they found non-trivial nonlinear surface waves of permanent form, whereas Hamilton *et al.* [9] gave a proof of the non-existence of non-trivial surface waves of permanent form. We show that the analyses given by these authors are flawed. The existence question therefore remains to be settled.

In Section 2, the governing equations are established and the main results from a linear analysis are summarized. In Section 3, the evolution equations, derived using the virtual work method proposed recently by Fu and Devenish [10], are given. The equations are solved numerically subject to appropriate initial conditions and, in Section 4, numerical results are presented for the case when there is initially only a single mode. In Section 5 the post-buckling problem is considered briefly with reference to the analysis in [8] and [9].

2. GOVERNING EQUATIONS AND LINEAR THEORY

We consider a homogeneous elastic half-space B composed of an isotropic material which possesses an initial unstressed state B_0. A pure homogeneous static deformation is then imposed upon B_0 to produce a finitely stressed equilibrium configuration denoted by B_e. The aim in this paper is to study the stability of B_e. To this end, a small amplitude time-dependent perturbation is superimposed on B_e, and the resulting (current) configuration is denoted by B_t. The position vectors of a representative particle relative to a common coordinate system are denoted by $X_A, x_i(X_A)$ and $\tilde{x}_i(X_A, t)$ in B_0, B_e and B_t respectively. We write

$$\tilde{x}_i(X_A, t) = x_i(X_A) + u_i(x_j, t), \tag{1}$$

where $u_i(x_j, t)$ is a small time-dependent displacement associated with the deformation $B_e \to B_t$. The half-space occupies the region $x_2 \le 0$ in B_e.

The deformation gradients arising from the deformations $B_0 \to B_t$ and $B_0 \to B_e$ are denoted by $\mathbf{F}$ and $\bar{\mathbf{F}}$ respectively and defined by

$$F_{iA} = \frac{\partial \tilde{x}_i}{\partial X_A}, \qquad \bar{F}_{iA} = \frac{\partial x_i}{\partial X_A}. \tag{2}$$

It is clear from (1) and (2) that $F_{iA} = (\delta_{ij} + u_{i,j})\bar{F}_{jA}$, where here and henceforth a comma indicates differentiation with respect to the implied spatial coordinate. The convention whereby upper case indices refer to coordinates in B_0 and lower case indices to coordinates in B_e will be observed.

In the absence of body forces, the equations of motion and the incompressibility constraint are given by

$$\pi_{iA,A} = \rho \ddot{u}_i, \qquad \det \mathbf{F} = 1, \tag{3}$$

where ρ is the (constant) density and $\boldsymbol{\pi}$ is the first Piola-Kirchhoff stress, whose component form is

$$\pi_{iA} = \frac{\partial W}{\partial F_{iA}} - pF^{-1}_{Ai}, \tag{4}$$

W being the strain-energy function (per unit volume), and p an arbitrary hydrostatic pressure. We denote by $\bar{p}$ the value of p in B_e.

Let $\mathbf{N}$ and $\mathbf{n}$ be the unit outward normals to the surfaces in B_0 and B_e, respectively. In the subsequent stability analysis, it will be assumed that the traction vector $\boldsymbol{\pi}\mathbf{N}$ prescribed on the surface of the half-space in B_0 is maintained during the incremental deformation $B_e \to B_t$. Such an assumption is usually referred to as a *dead-load* traction boundary condition and is represented by

$$(\pi_{iA} - \bar{\pi}_{iA})N_A = 0 \quad \text{on } X_2 = 0, \tag{5}$$

where $\bar{\pi}_{iA}$ is the value of π_{iA} calculated from (4) with $\mathbf{F}$ replaced by $\bar{\mathbf{F}}$ and p by $\bar{p}$.

By introducing a tensor function with components χ_{ij}, (3) and (5) may be written

$$\chi_{ij,j} = \rho \ddot{u}_i, \quad \chi_{ij} n_j = 0 \tag{6}$$

in $x_2 < 0$ and on $x_2 = 0$ respectively, where, as in Fu and Ogden [11],

$$\chi_{ij} = \mathcal{A}^1_{0jilk} u_{k,l} + \frac{1}{2}\mathcal{A}^2_{0jilknm} u_{k,l} u_{m,n} + \bar{p}(u_{j,i} - u_{j,k}u_{k,i}) - p^*(\delta_{ji} - u_{j,i}) + O(\epsilon^3). \tag{7}$$

In (7), $\mathcal{A}^1_{0jilk}$ and $\mathcal{A}^2_{0jilknm}$ are the first- and second-order instantaneous elastic moduli, whose expressions in terms of principal stretches can be found in [1] or [11], p^* is the incremental pressure associated with the deformation $B_e \to B_t$, so that

$$p = \bar{p} + p^*, \tag{8}$$

and ϵ is a small parameter characterizing the amplitude of p^* and $u_{i,j}$.

On expanding the constraint equation $(3)_2$, we obtain

$$u_{i,i} = \frac{1}{2} u_{m,n} u_{n,m} + O(\epsilon^3). \tag{9}$$

Equations $(6)_1$ and (9) are the nonlinear equations governing the incremental displacement u_i and pressure p^* in $x_2 < 0$, and $(6)_2$ must be satisfied on $x_2 = 0$.

Before proceeding further, we first non-dimensionalize the governing equations and boundary conditions using L (to be defined) as the length scale (for u_i, x_i), μ (to be defined) as the stress scale (for $\bar{p}, p^*, \mathcal{A}^1_{0jilk}$ etc.) and $L\sqrt{\rho/\mu}$ as the time scale (for t). In order to avoid introducing additional notation, we shall use the same symbols to denote the corresponding non-dimensionalized quantities. The non-dimensionalized forms of the governing equations and boundary conditions then remain unchanged except that $\rho = 1$.

¿From now on we assume that the incremental deformation is one of plane strain (i.e. $u_3 \equiv 0$ with u_1 and u_2 independent of x_3) and that the principal axes of strain in B_e are aligned with the coordinate axes. Linearizing (9) and (6), we obtain

$$u_{i,i} = 0, \tag{10}$$

$$\mathcal{A}^1_{0jilk} u_{k,lj} - p^*_{,i} = \ddot{u}_i, \tag{11}$$

$$\mathcal{A}^1_{02ilk} u_{k,l} + \bar{p} u_{2,i} - p^* \delta_{2i} = 0, \quad \text{on} \quad x_2 = 0. \tag{12}$$

Equation (10) implies the existence of a 'stream function' ψ such that

$$u_1 = \psi_{,2}, \qquad u_2 = -\psi_{,1}. \tag{13}$$

On eliminating p^* from (11) and (12) and then expressing the resulting equations in terms of ψ, we obtain [3]

$$\alpha \psi_{,1111} + 2\beta \psi_{,1122} + \gamma \psi_{,2222} = \nabla^2 \psi_{,tt}, \tag{14}$$

$$\left. \begin{array}{l} \gamma \psi_{,22} - (\gamma - \sigma_2) \psi_{,11} = 0, \\ \gamma \psi_{,222} + (\gamma + 2\beta - \sigma_2) \psi_{,112} - \psi_{,2tt} = 0, \end{array} \right\} \quad \text{on} \ \ x_2 = 0, \tag{15}$$

where σ_2 is the principal Cauchy stress in B_e along the x_2-axis and

$$\alpha = \mathcal{A}^1_{01212}, \quad \gamma = \mathcal{A}^1_{02121}, \quad 2\beta = \mathcal{A}^1_{01111} + \mathcal{A}^1_{02222} - 2\mathcal{A}^1_{01122} - 2\mathcal{A}^1_{01221}.$$

The eigenvalue problem (14)-(15) admits a Rayleigh wave solution of the form

$$\psi = H(x_2) \mathrm{e}^{ik(x_1 - vt)}, \tag{16}$$

where k is the wavenumber, v is the wavespeed and $H(x_2)$ is the shape function. This form of solution has been studied in detail by Dowaikh and Ogden [3], and their main results are now summarized in preparation for the nonlinear analysis.

The Rayleigh wavespeed v is determined by

$$\gamma(\alpha - v^2) - (\gamma - \sigma_2)^2 = (v^2 - 2\gamma + 2\sigma_2 - 2\beta)\sqrt{\gamma(\alpha - v^2)}, \tag{17}$$

which, on setting $v = 0$, gives the bifurcation condition for neutral stability, namely

$$\alpha\gamma - (\gamma - \sigma_2)^2 = -2(\gamma - \sigma_2 + \beta)\sqrt{\alpha\gamma}. \tag{18}$$

Since α, β, γ are functions of the principal stretches and σ_2 is related to $\bar{p}$, this condition relates $\bar{p}$ to the principal stretches for which neutral stability occurs.

When $v = 0$, the shape function $H(x_2)$ is given by

$$H(x_2) = \frac{1}{|k|}\left(\xi_1 \mathrm{e}^{s_1|k|x_2} + \xi_2 \mathrm{e}^{s_2|k|x_2}\right), \tag{19}$$

where

$$\xi_1 = \frac{\gamma s_2^2 + \gamma - \sigma_2}{\gamma(s_2^2 - s_1^2)}, \qquad \xi_2 = \frac{\gamma s_1^2 + \gamma - \sigma_2}{\gamma(s_1^2 - s_2^2)},$$

and s_1, s_2 are the roots with positive real part of $s^2 = \gamma^{-1}\left(\beta \pm \sqrt{\beta^2 - \alpha\gamma}\right)$. In the degenerate case $s_1 = s_2$, (19) is not valid but the appropriate shape function can be obtained by taking the limit $s_2 \to s_1$ in (19) and using l'Hopital's rule.

On substituting (19) and (16) into (13), we obtain

$$u_1 = \xi_m s_m e^{s_m|k|x_2} e^{ikx_1} \equiv U_1(x_2, k)e^{ikx_1}, \tag{20}$$

$$u_2 = -i(k/|k|)\xi_m e^{s_m|k|x_2} e^{ikx_1} \equiv U_2(x_2, k)e^{ikx_1}, \tag{21}$$

where we have employed a modified summation convention in which a suffix appearing more than once is summed from 1 to 2. This convention will be observed in the subsequent analysis.

The corresponding incremental pressure can be determined by substituting $p^* = P(x_2, k)\mathrm{e}^{ikx_1}$ into (11) with $i = 1$ and $\partial/\partial t = 0$. This gives

$$p^* = -ik\xi_m F(s_m)e^{s_m|k|x_2} e^{ikx_1} \equiv P(x_2, k)e^{ikx_1}, \tag{22}$$

where

$$F(s_m) = \gamma s_m^3 + (v^2 + \mathcal{A}^1_{01122} + \mathcal{A}^1_{02112} - \mathcal{A}^1_{01111})s_m. \tag{23}$$

Although we have assumed a sinusoidal wave solution (16), we remark that equations (17) and (18) could have been obtained by adopting a more general form of solution such as that considered in [2]. Thus, neither the amplitude nor the mode of instability is determined by a linear analysis.

3. NONLINEAR ANALYSIS

Henceforth, we assume that B_e is also a state of plane strain and we write the principal stretches as $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$ so that B_e is characterized by two parameters, namely λ and $\bar{p}$.

Let $\lambda = \lambda_0$, $\bar{p} = \bar{p}_0$ be a solution of the bifurcation condition (18) and let

$$\lambda = \lambda_0 + \epsilon\lambda_1, \qquad \bar{p} = \bar{p}_0 + \epsilon\bar{p}_1 \tag{24}$$

be the values of λ and $\bar{p}$ in B_e, which is now assumed to be a near-neutral configuration. Suppose, for example, a perturbation in the form of a single mode is excited in this near-neutral configuration. We aim to determine how such a mode will evolve. The formulation is also valid for the case when more than one mode is excited initially provided the modes are harmonics of a primary mode.

For near-neutral non-dispersive modes, the acceleration term $\ddot{u}_m$ is required to balance the quadratic terms. From this we deduce that the appropriate modulation time scale is defined by $\tau = \epsilon^{1/2}t$. It is known that for non-dispersive waves, a single mode will excite all the harmonics even at leading order because of nonlinear interaction. Thus, the nonlinear solution takes the form

$$u_n = \epsilon \sum_m A_m(\tau) U_n(x_2, m) e^{imx_1}, \quad n = 1, 2, \tag{25}$$

$$p^* = \epsilon \sum_m A_m(\tau) P(x_2, m) e^{imx_1}, \tag{26}$$

where $A_m(\tau)$ is the amplitude of the m-th harmonic and here and henceforth the indices under the summation signs range from $-\infty$ to ∞ excluding zero. The form of solutions (25) and (26) is appropriate to perturbations which are periodic in x_1. We have taken the wavenumber of the fundamental mode in (25) and (26) to be unity, which corresponds to taking the lengthscale L in the non-dimensionalization to be the inverse of the fundamental wavenumber. For perturbations which are not periodic, (25) and (26) should be replaced by Fourier integrals. At any stage in the subsequent analysis, conversion of sums to integrals to describe non-periodic perturbations is straightforward.

The evolution equations for the amplitudes $A_m(\tau)$ of various Fourier modes may be derived using the virtual work method of Fu and Devenish [10]. This uses the line integral

$$I = \oint_C \chi_{ij} n_j \hat{u}_i dS, \tag{27}$$

where n_j is the outward normal to the path, $\hat{u}_i$ is a 2π-periodic field given by $\hat{u}_n = U_n(x_2, -k)\mathrm{e}^{-ikx_1}$, where k is an arbitrary integer, and C is the boundary of the rectangular region $S = [0 \leq x_1 \leq 2\pi,\ -h \leq x_2 \leq 0]$, where h is a positive constant. In the limit $h \to \infty$, we have $I \to 0$ since (i) from $(6)_2$ the integrand vanishes on $x_2 = 0$, (ii) the integrand tends to zero as $x_2 \to \infty$, and (iii) the integrals on the two vertical paths cancel because of the periodicity of the integrand. On use of the divergence theorem, $(6)_1$ and the scaling $\tau = \epsilon^{1/2}t$, equation (27) may be written

$$I = \int_S (\chi_{ij,j}\hat{u}_i + \chi_{ij}\hat{u}_{i,j}) dx_1 dx_2 = \int_S \left(\epsilon \frac{\partial^2 u_i}{\partial \tau^2} \hat{u}_i + \chi_{ij}\hat{u}_{i,j} \right) dx_1 dx_2. \tag{28}$$

The evolution equations are obtained by substituting (25) and (26) into this equation and then taking the limit $h \to 0$. The detailed derivation follows that of [10] which treated a related problem. It can be shown that the evolution equations are

$$c_0 \frac{d^2 A_k}{d\tau^2} = c_1 k^2 A_k + ik^2 \sum_{k'} \mathcal{K}(k, k') A_{k'} A_{k-k'} \quad (A_{-k} = A_k^*, \quad k = 1, 2, 3, \ldots), \tag{29}$$

where the asterisk signifies complex conjugate,

$$c_0 = -\frac{1}{2}\left\{\frac{1+s_1^2}{s_1}\xi_1^2 + \frac{1+s_2^2}{s_2}\xi_2^2 + \frac{4\xi_1\xi_2}{s_1+s_2}(1+s_1s_2)\right\}, \tag{30}$$

$$c_1 = -\frac{\xi_a\xi_b}{s_a+s_b}\left\{\lambda_1\hat{\mathcal{A}}^1_{0nmqp}\Gamma(p,q,k,a)\Gamma(m,n,-k,b) + \bar{p}_1\Gamma(p,q,k,a)\Gamma(q,p,-k,b)\right\}, \tag{31}$$

$$\mathcal{K}(k,k') = \frac{\xi_a\xi_b\xi_c|k'||k-k'|}{2(s_a|k|+s_b|k'|+s_c|k-k'|)}\Big\{\mathcal{A}^2_{0qpnmsr}\Gamma(p,q,-k,a)\times$$

$$\Gamma(r,s,k',b)\Gamma(m,n,k-k',c) + \frac{2k'}{|k'|}F(s_b)\Gamma(n,m,-k,a)\Gamma(m,n,k-k',c)$$

$$-\frac{k}{|k|}F(s_a)\Gamma(n,m,k',b)\Gamma(m,n,k-k',c)\Big\}, \tag{32}$$

the function Γ being defined through

$$\Gamma(a,b,k,m) = (\delta_{2a}k/|k| + is_m\delta_{1a})(s_m\delta_{2b} + i\delta_{1b}k/|k|). \tag{33}$$

The moduli in (32) are evaluated for $\lambda = \lambda_0$ and

$$\hat{\mathcal{A}}^1_{0jinm} = \left(\frac{\partial}{\partial\lambda}\mathcal{A}^1_{0jinm}\right)\Big|_{\lambda=\lambda_0}. \tag{34}$$

When the nonlinear terms are neglected, we deduce from the reduced form of (29) that the linear Rayleigh wavespeed is given by $v^2 = -(c_1/c_0)\epsilon$. On the other hand, with λ and $\bar{p}$ given by (24), a leading order asymptotic expression for v^2 may also be obtained from (17). This provides a check on the expressions for c_0 and c_1.

4. DYNAMIC SOLUTIONS

In this section we describe the numerical solution of the system of equations (29). We assume that the pre-stress is a uniaxial tension or compression in the x_1-direction ($\sigma_2 = 0$) and that the strain-energy function is given by

$$W = 2\mu(\lambda_1^m + \lambda_2^m - 2)/m^2, \tag{35}$$

where μ is the shear modulus and m is a real constant. With μ taken to be the μ used in the non-dimensionalization, it does not feature in the analysis. Three values of m often used in the literature are $m = 2, 1, 1/2$. The case $m = 2$ is associated with the neo-Hookean strain-energy function, whilst $m = 1$ leads to the degenerate case $s_1 = s_2$ mentioned in Section 2. To simplify the analysis and to allow for material nonlinearity, we take $m = 1/2$. The bifurcation condition (18) then reduces to $\lambda^3 - 3\lambda^2 - 2\lambda + 2 = 0$ [3], which has positive real roots $\lambda_0 = 0.5858$, 3.4142. In the following calculations, we take $\lambda_0 = 0.5858$.

The system of equations (29) is first replaced by a finite system by truncation at $k' = \pm N$, where N is sufficiently large for the results to be essentially independent

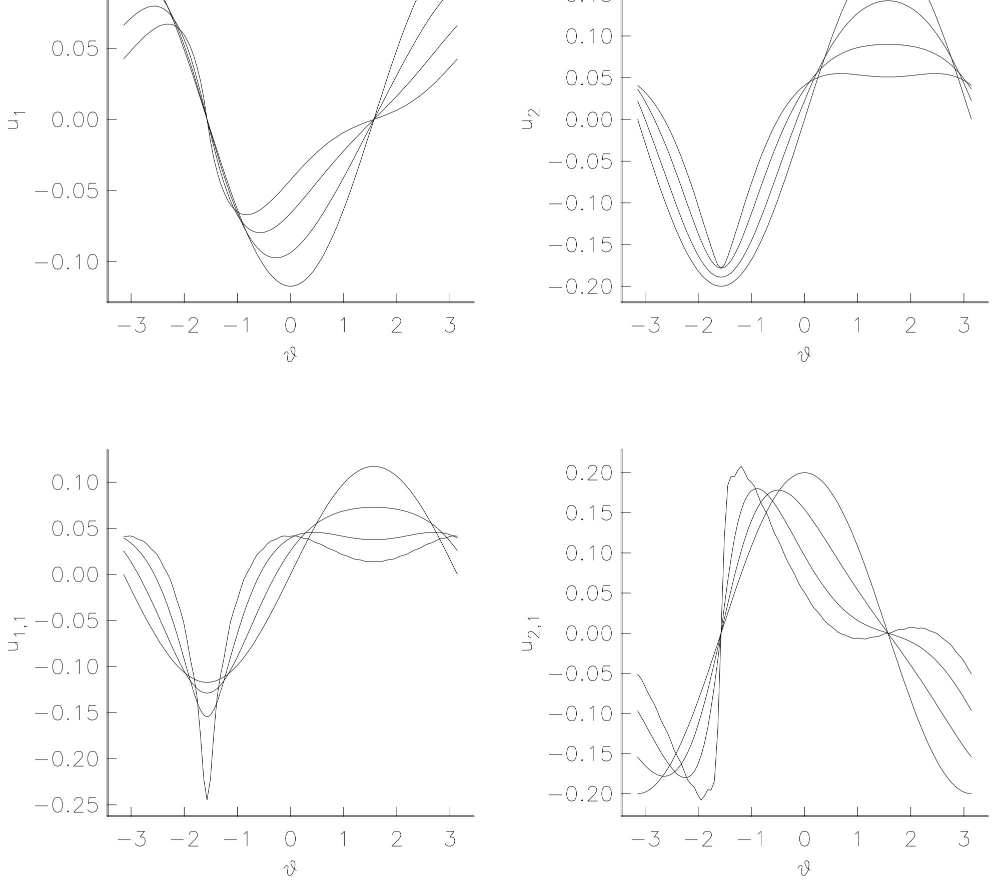

Fig. 1. Plots of $u_1, u_2, u_{1,1}$ and $u_{2,1}$ for $\tau = 0, 2, 3, 3.63$ and $c_1/c_0 = -0.1$. Profiles for larger values of τ are steeper at the shock position.

of the choice of N. The finite system is then integrated by a Runge-Kutta method with step-length self-adjusted. We focus on the initial conditions

$$A_k(0) = 0.1\delta_{1k}, \qquad dA_k(0)/d\tau = 0, \qquad k = 1, 2, 3, \ldots. \tag{36}$$

So, only the fundamental mode exists initially. We have taken $A_1(0) = 0.1$ but we note that other (complex) values having the same magnitude will yield results which differ only by a constant phase shift since we can scale A_k by $e^{ik\phi}$, where ϕ is a constant, without affecting the form of (29). Also, we can fix the amplitude $A_1(0)$ since any variation in $A_1(0)$ can be absorbed by scaling τ and varying c_1. Thus, we may fix the initial conditions as (36) and investigate the effect of varying c_1/c_0.

We first take $c_1/c_0 = \pm 0.1$. Numerical experimentation shows that $N = 25$ is adequate, with higher values of N giving the same results except near the shock formation time. Figures 1 and 2 show the evolution of u_1, u_2, $u_{1,1}$ and $u_{2,1}$ for $c_1/c_0 = -0.1$ and 0.1, respectively. We note that when $c_1/c_0 = -0.1$ the half-space is sub-critically near-neutral (and the linear theory predicts $A_1 = 0.1\cos(\sqrt{0.1}\tau)$), while when $c_1/c_0 = 0.1$ the half space is super-critically near-neutral (and linear theory predicts $A_1 = 0.1\cosh(\sqrt{0.1}\tau)$). In each case, nonlinear modulation leads to the formation of shocks in the profile of $u_{2,1}$ and spikes in the profile of $u_{1,1}$. The super-critical case $c_1/c_0 = 0.1$ gives an earlier shock formation time and larger shock amplitudes.

To show how energy is transferred to higher modes through nonlinear interaction, the evolution of A_1, A_2, A_3 and A_{25} up to the shock formation time is plotted in Fig. 3 for the two cases shown in Figs 1 and 2. The results for A_1 given by the linear theory

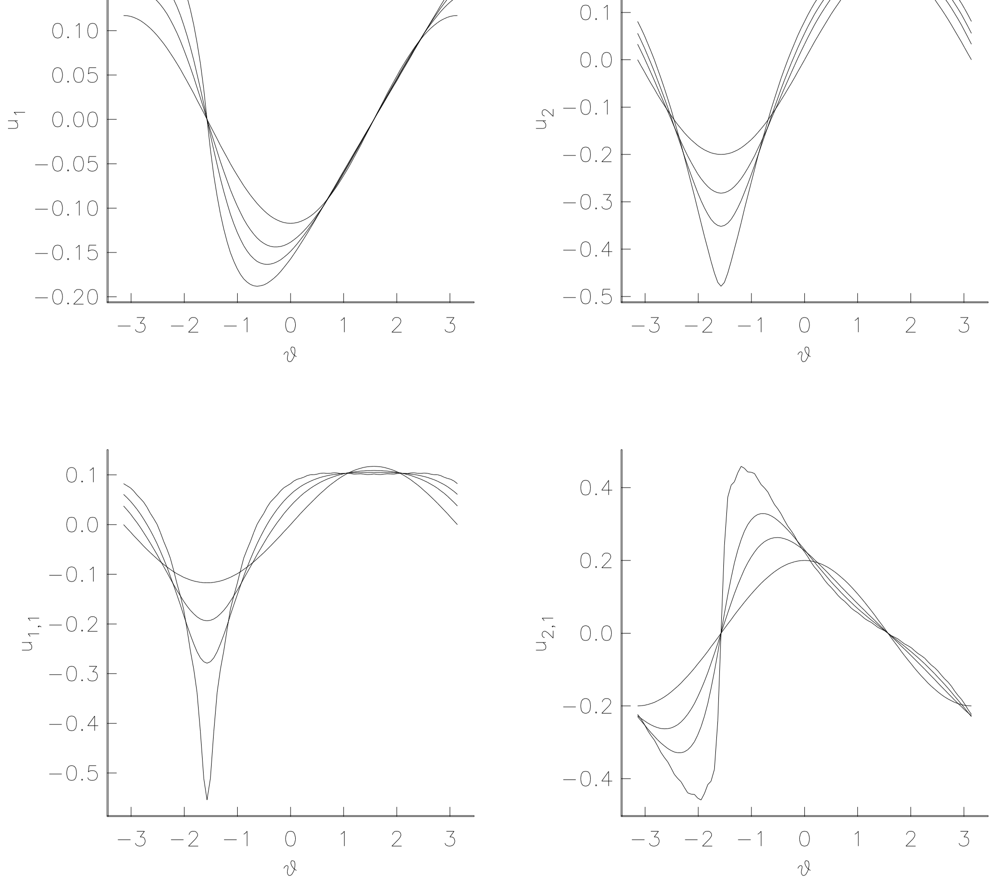

Fig. 2. Plots of $u_1, u_2, u_{1,1}$ and $u_{2,1}$ for $\tau = 0, 1.9, 2.5, 2.93$ and $c_1/c_0 = 0.1$. Profiles for larger values of τ are steeper at the shock position.

are shown as dotted lines. In each case higher harmonics are continually generated and grow monotonically. An important feature is that, despite the generation of higher harmonics, the fundamental mode is also amplified by nonlinear effects. In this sense nonlinear effects are destabilizing, but we observe that amplification by nonlinear effects is almost negligible. Shocks form before any significant growth has occurred. The main effect of nonlinearity is to generate higher harmonics and lead to the formation of shocks. Further calculations show that these conclusions are also valid for other positive values of c_1/c_0.

As c_1/c_0 decreases gradually from -0.1, it takes longer and longer for nonlinear effects to become pronounced. There is then time for more higher harmonics to be excited, and larger values of the truncation number N are needed for larger values of $|c_1/c_0|$. As expected, shock formation time increases with decreasing c_1/c_0 (which corresponds to departure from the near-neutral regime into the non-neutral regime for which the time scale is longer). Figs 4(a,b) show the evolution of various harmonics for $c_1/c_0 = -0.2$ and -0.3 respectively, where we have taken $N = 50$. Again, there is negligible deviation of the nonlinear from the linear A_1 before shock formation occurs.

We conclude that for all values of c_1/c_0 (positive or negative), the main effect of nonlinearity is to excite higher harmonics and to lead to the formation of shocks. The fundamental mode is little affected by nonlinear interaction. Nonlinear effects become increasingly weaker as c_1/c_0 decreases from zero, resulting in longer shock formation times.

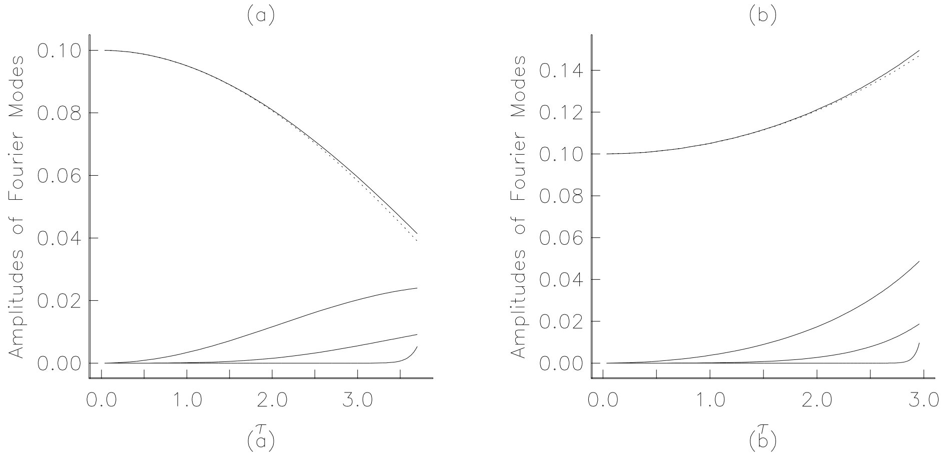

Fig. 3. Plots of A_1, A_2, A_3 and $100A_{25}$ against τ for (a) $c_1/c_0 = -0.1$ and (b) $c_1/c_0 = 0.1$. Higher harmonics have smaller amplitudes. Dotted lines are linear results for A_1.

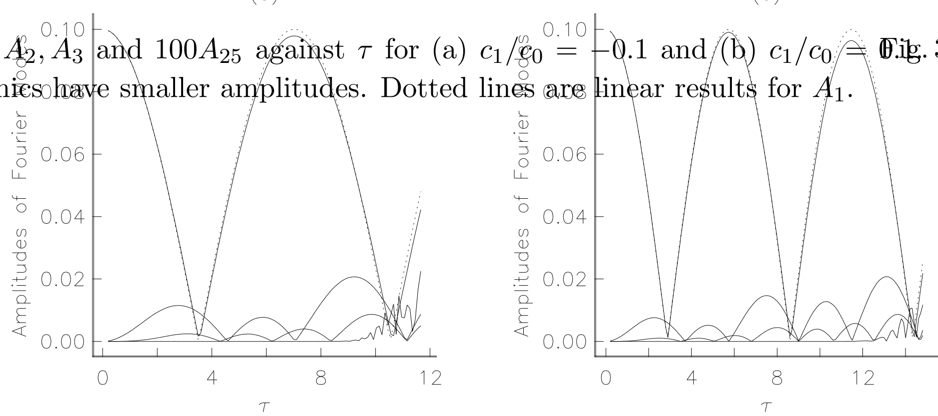

Fig. 4. Plots of A_1, A_2, A_3 and $100A_{50}$ against τ for (a) $c_1/c_0 = -0.2$ and (b) $c_1/c_0 = 0.3$. Higher harmonics have smaller amplitudes. Dotted lines are linear results for A_1.

5. STATIC SOLUTIONS

We now proceed to the determination of post-buckling states, which, if they exist, correspond to static solutions of the evolution equations (29). They are governed by

$$c_1 A_k + i \sum_{k'} \mathcal{K}(k,k') A_{k'} A_{k-k'} = 0 \quad (A_{-k} = A_k^*, \quad k = 1, 2, 3, \ldots). \tag{37}$$

The form of these equations suggests that a possible class of solutions is given by $A_k = -iB_k$ $(k = 1, 2, \ldots)$, with B_k real. In terms of B_k, (37) becomes

$$c_1 B_k + \sum_{k'} \mathcal{K}(k,k') B_{k'} B_{k-k'} = 0, \tag{38}$$

and we now have $B_{-k} = -B_k$ $(k = 1, 2, \ldots)$. Systems of algebraic equations of this form have been discussed in [8] and [9] with conflicting conclusions. Whereas in [8] it was claimed that non-trivial solutions could be found, it was shown in [9] that the system of equations can have no non-trivial solutions. The methods used in [8] and [9] are now discussed in the context of the present problem.

5.1 The Method of Parker and Talbot [8]

Following [8], we first truncate the system (38) at $k' = \pm 3$ and obtain three equations for B_1, B_2 and B_3 which can be solved exactly to yield four sets of real solutions. We may use each of these solutions as a starting solution and increase the truncation number gradually. After solution of N equations for $B_1, B_2, \cdots, B_N$, we solve $N+1$ equations with the initial guess for the first N unknowns taken as the solution in the previous calculation and B_{N+1} set to zero. At each step the system of equations is solved using Nag Library subroutine C05NBF, and the progression stops when the solution converges. We find that the solution always converges to the trivial solution if the truncation number is increased in unit steps. Since solutions were obtained in [8] with the truncation number increased with a step of 3, we also used this step for (38). Non-trivial convergent solutions were indeed found. Doubt is therefore cast on the validity of solutions obtained with truncation number steps other than unity. The simple equation $H(\theta)+H(\theta)^2=0$ illustrates the problem. It has only two solutions, $H=0$ and a non-trivial solution $H=-1$. However, if we substitute

$$H=\sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{in\theta}, \quad A_{-n}=A_n,$$

into the equation, we obtain the infinite system

$$A_n+\sum_{m=-\infty}^{\infty} A_m A_{n-m}=0, \quad n=0,1,2,\ldots. \tag{39}$$

With the procedure explained above and with the truncation number increased in steps of unity, only the correct solutions are obtained, but if steps not equal to unity (including 3) are used, spurious solutions can be found. Such solutions are continuous but their derivatives suffer discontinuities (interestingly, the non-trivial solutions found in [8] and the solutions we found for (38) also have this feature). Thus, in general, solutions of equations of the form (38) obtained with the truncation number increased in steps greater than unity may not be the correct solutions.

5.2 The Method of Hamilton *et al.* [9]

Using the property $\mathcal{K}(k,k-k')=\mathcal{K}(k,k')$ and truncating the system (38) at $k'=\pm N$, we obtain, after some manipulation,

$$c_1 B_k+\sum_{k'=1}^{k-1}\mathcal{K}(k,k')B_{k'}B_{k-k'}-2\sum_{k'=k+1}^{N}\mathcal{K}(k,k')B_{k'}B_{k'-k}=0. \tag{40}$$

For perturbations requiring representation in terms of a continuous spectrum, (40) is replaced by

$$c_1 B(k)+\int_0^k \mathcal{K}(k,k')B(k')B(k-k')dk'-2\int_k^{k_{max}}\mathcal{K}(k,k')B(k')B(k'-k)dk'=0, \tag{41}$$

where k_{max} plays a role analogous to N. Following [9], we make the substitutions

$$k'=k_{max}y', \quad k=k_{max}y, \quad f(y)=k_{max}^2 B(k_{max}y).$$

We then have $B(k)=k_{max}^{-2}f(k/k_{max})$, and (41) gives

$$c_1 f(y)+\int_0^y \mathcal{K}(y,y')f(y')f(y-y')dy'-2\int_y^1 \mathcal{K}(y,y')f(y')f(y'-y)dy'=0, \tag{42}$$

where we have used the property $\mathcal{K}(ak, ak') = a\mathcal{K}(k, k')$ deduced from (32). According to the argument given in [9], since the solution for $f(y)$ obtained by solving (42) is independent of k_{max}, $B(k) \to 0$ as $k_{max} \to \infty$, and this shows that any solution of (41) converges to the trivial solution as $k_{max} \to \infty$. Since (40) is analogous to (41), it can be deduced further that the system (40) has no non-trivial solutions in the limit $N \to \infty$ either. Thus, if the argument in [9] were correct, the pre-stressed half-space would not admit any post-buckling states.

However, the argument is flawed. To show this, we consider the infinite system

$$A_k = \frac{1}{k^2} \sum_{k'=1}^{\infty} (k - k')^2 A_{k'} A_{k-k'}, \qquad k = 1, 2, 3, \ldots \tag{43}$$

This has a solution given by $A_k = 6/\pi^2 k^2 \, (k = 1, 2, \ldots)$, but an application of the argument in [9] would lead to the conclusion that (43) can have no non-trivial solutions.

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