

BUCKLING OF AN ELASTIC HALF-SPACE WITH SURFACE IMPERFECTIONS[†]

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Abstract. *Over the years, linear analysis of the stability of pre-stressed elastic half-spaces has been carried out for various kinds of materials and various forms of pre-stress; see Biot (1965), Nowinski (1969a, b), Willson (1973a, b), Usmani & Beatty (1974), Chadwick & Jarvis (1979), Wu (1979, 1980), Reddy (1983), Ogden (1984), Wu & Cao (1983, 1984), and Dowaikh & Ogden (1990, 1991). Furthermore, Wu & Cao (1984) showed that the bifurcation condition for an elastic half-space is in fact the same as those for an infinite space with a crack of arbitrary shape, for a circular disk, and for a semi-infinite strip. A linear stability analysis provides the condition under which a pre-stressed half-space may buckle, but it gives no information about the form or stability of post-buckling states; such information can only be found by a nonlinear analysis. For a pre-stressed half-space, the critical stress for marginal stability is independent of mode numbers, which makes the corresponding weakly nonlinear analysis very different from that for problems where there is usually a preferred buckling mode. A first attempt at finding a post-buckling solution was made by Ogden & Fu (1996) who looked for a solution in the form of a Fourier summation. Imposition of a solvability condition at second order of a successive approximation yielded an infinite system of quadratic equations for the Fourier amplitudes. However no non-trivial solutions were found for such a system of algebraic equations. To shed some light on this open problem, we consider in this paper the stability of an imperfect elastic half-space, a half-space the surface of which is not flat but has an sinusoidal profile (a modal imperfection). The imperfect half-space is subjected to a uni-axial compression and the evolution of the surface elevation is followed as the compression is increased. It is found that as the compression approaches a critical value, which is smaller than the critical value predicted by the linear theory for a perfect half-space, static shocks begin to develop in the profiles of surface elevation; no stable solutions exist beyond this critical value. These results support the conjecture that post-buckling solutions associated with a pre-stressed half-space without imperfections may contain static shocks (i.e. singular surfaces across which some of the deformation gradients are discontinuous).*

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1 FORMULATION OF THE PROBLEM

We consider the deformation of an incompressible elastic half-space the accessible surface of which has a periodic profile initially. Relative to a rectangular coordinate system, the half-space in its un-deformed configuration is given by

$$-\infty < X_1, X_3 < \infty, \quad a\eta(X_1/L) < X_2 < \infty,$$

where a and L are the amplitude and wavelength of the initial surface profile and the function $\eta(X_1/L)$ will be specified later. This imperfect half-space is subsequently subjected to a uni-axial compression and the deformation is denoted by

$$\tilde{x}_i = \tilde{x}_i(X_A), \tag{1}$$

where (\tilde{x}_i) are the coordinates in the deformed configuration of the same material particle which has coordinates (X_A) in the undeformed configuration. The governing equations are the equilibrium equations, constitutive equations and the incompressibility condition given by

$$\pi_{iA,A} = 0, \quad \pi_{iA} = \frac{\partial W}{\partial F_{iA}} - pF_{Ai}^{-1}, \quad \det(\mathbf{F}) = 1, \tag{2}$$

where (π_{iA}) is the first Piola-Kirchhoff stress tensor, $\mathbf{F} = (F_{iA}) = (\partial\tilde{x}_i/\partial X_A)$ is the deformation gradient tensor, W the strain energy function and p the pressure corresponding to the constraint of incompressibility.

The curved surface is assumed to be stress free. This implies

$$\pi_{iA}N_A = 0 \quad \text{on } X_2 = a\eta(X_1/L), \tag{3}$$

where (N_A) is the unit outward normal to the surface. The uni-axial compression along the X_1 -direction is represented by the condition

$$\pi_{iA}N_A = -TN_A \tag{4}$$

on any surface $X_1 = \text{const.}$, where $T > 0$ is the compression.

Before we proceed further, it is convenient to non-dimensionalize the problem by scaling π_{iA} , p , T and W by μ , X_A and \tilde{x}_i by L^* where μ is the shear modulus of the composing material when it is stress free and L^* is arbitrary subject to the condition that it is of the same order as L (the reason that we do not use L as the lengthscale is that by choosing L^* appropriately we can make the wavelength of the initial surface elevation unit in terms of the new reference coordinates to be introduced later). With the same symbols used for the scaled quantities, the non-dimensionalized forms of (2) and (4) remains the same but the stress free boundary condition (3) is replaced by

$$\pi_{iA}N_A = 0 \quad \text{on } X_2 = \epsilon^2\eta(bX_1), \tag{5}$$

where $\epsilon^2 = a/L^*$, $b = L^*/L$. We assume that ϵ is a small parameter and η is a periodic function with an $O(1)$ period. This implies that the gradient of the initial surface elevation is $O(\epsilon^2)$. In this paper we consider deformations the gradients of which are $O(1)$.

Since the normal (N_1, N_2) is parallel to $(-\epsilon^2 b\eta'(bX_1), 1)$, it follows that

$$N_A = \delta_{2A} - \epsilon^2 b\eta'(bX_1)\delta_{1A} + O(\epsilon^4). \quad (6)$$

On substituting (6) into (5) and expanding π_{iA} about $X_2 = 0$, we obtain

$$\pi_{i2} + \epsilon^2 \{ \pi_{i2,2}\eta(bX_1) - b\eta'(bX_1)\pi_{i1} \} + O(\epsilon^4) = 0, \quad \text{on } X_2 = 0. \quad (7)$$

The $O(1)$ problem with $\epsilon \equiv 0$ is clearly the problem of a perfect half-space subjected to a uni-axial compression. We assume that the homogeneous solution is unique and is denoted by $\tilde{x}_i = x_i(X_A)$ where

$$x_1 = \lambda X_1, \quad x_2 = \lambda^{-1} X_2, \quad x_3 = X_3, \quad (8)$$

and λ is the principal stretch along the X_1 -direction. We denote the deformation gradient tensor corresponding to (8) by $\bar{\mathbf{F}}$ and the pressure by \bar{p} which is related to $\bar{\mathbf{F}}$ by applying the stress free boundary condition $\bar{\pi}_{i2} = 0$ on $X_2 = 0$ where $\bar{\pi}_{iA}$ is the value of π_{iA} with (\mathbf{F}, p) replaced by $(\bar{\mathbf{F}}, \bar{p})$. To determine the solution with $\epsilon \neq 0$, it is convenient to subtract out the homogeneous solution by defining a tensor function χ_{ij} through

$$\chi_{ij} = (\pi_{iA} - \bar{\pi}_{iA})\bar{F}_{jA} \quad (9)$$

and an incremental displacement field \mathbf{u} and pressure field p^* through

$$\mathbf{u} = \tilde{\mathbf{x}} - \mathbf{x}, \quad p^* = p - \bar{p}. \quad (10)$$

We will look for a solution for (\mathbf{u}, p^*) which is of order ϵ . We note that such a solution is a bifurcation solution modified slightly by the surface imperfection and is not a solution forced by the imperfection (a forced solution is only of order ϵ^2).

In terms of χ_{ij} and u_i , the equilibrium equation (2a) and the incompressibility condition (2c) become

$$\chi_{ij,j} = 0, \quad u_{i,i} = \frac{1}{2}u_{m,n}u_{n,m} + O(\epsilon^3), \quad (11)$$

where here and hereafter a comma denotes differentiation with respect to the spatial coordinates (x_n) . By taking $b = \lambda$, the boundary condition (7) becomes

$$\chi_{ij}n_j = \epsilon^2\eta'(x_1)\bar{\pi}_{i1} + O(\epsilon^3), \quad \text{on } x_2 = 0. \quad (12)$$

We may expand the right hand side of the constitutive equation (2) about $(\bar{\mathbf{F}}, \bar{p})$. It can be shown (see e.g. Fu & Ogden 1999) that this leads to

$$\chi_{ij} = \mathcal{A}_{jilk}^1 u_{k,l} + \frac{1}{2}\mathcal{A}_{jilknm}^2 u_{k,l}u_{m,n} + \bar{p}(u_{j,i} - u_{j,k}u_{k,i}) - p^*(\delta_{ji} - u_{j,i}) + O(\epsilon^3), \quad (13)$$

where the tensors \mathcal{A}^1 and \mathcal{A}^2 are the first- and second-order tensors of instantaneous elastic moduli defined by

$$\mathcal{A}_{jilk}^1 = \bar{F}_{jA}\bar{F}_{lB} \left. \frac{\partial^2 W}{\partial F_{iA}\partial F_{kB}} \right|_{\mathbf{F}=\bar{\mathbf{F}}}, \quad \mathcal{A}_{jilknm}^2 = \bar{F}_{jA}\bar{F}_{lB}\bar{F}_{nC} \left. \frac{\partial^3 W}{\partial F_{iA}\partial F_{kB}\partial F_{mC}} \right|_{\mathbf{F}=\bar{\mathbf{F}}}. \quad (14)$$

When (13) is substituted into (11a), an identity derived from $(F_{Ai}^{-1})_{,A} \equiv 0$ can be used to simplify the resulting expression. We have

$$\mathcal{A}_{jilk}^1 u_{k,lj} + \mathcal{A}_{jilknm}^2 u_{m,n} u_{k,lj} - p_{,j}^* (\delta_{ji} - u_{j,i}) + O(\epsilon^3) = 0. \quad (15)$$

Our problem is then to solve (11b) and (15) subject to the boundary condition (12) and the decay condition $\mathbf{u}, p^* \rightarrow 0$ as $x_2 \rightarrow -\infty$. With the right hand side of (12) replaced by zero, this problem reduces to the problem studied by Ogden & Fu (1996) for a perfect half-space.

2 AMPLITUDE EQUATIONS

We now assume that initially the scaled surface elevation is given by

$$\eta(x_1) = \cos x_1.$$

The boundary condition (12) then becomes

$$\chi_{ij} n_j = \frac{i}{2} \epsilon^2 \bar{\pi}_{i1} (e^{ix_1} - e^{-ix_1}) + O(\epsilon^3), \quad \text{on } x_2 = 0. \quad (16)$$

We look for an asymptotic solution of the form

$$u_i = \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \dots, \quad p^* = \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \dots. \quad (17)$$

We also expand λ and \bar{p} as

$$\lambda = \lambda_0 + \epsilon \lambda_1, \quad \bar{p} = \bar{p}_0 + \epsilon \bar{p}_1, \quad (18)$$

where \bar{p}_0 is related to λ_0 by satisfying the stress free boundary condition in the $O(1)$ problem. On substituting (17) and (18) into the governing equations (11b) and (15) and the boundary condition (12) and equating the coefficients of ϵ , we find that $u_i^{(1)}$ and $p^{(1)}$ satisfy the same eigenvalue problem as their counterparts in Ogden & Fu (1996). This eigenvalue problem determines the bifurcation condition, satisfied by (λ_0, \bar{p}_0) , under which the perfect half-space is marginally stable. It also determines the shape functions for the normal modes. For the present non-dispersive problem, we anticipate that the $O(\epsilon^2)$ problem for $(u_i^{(2)}, p^{(2)})$ can not be solved unless a solvability condition is imposed on the leading order solution $(u_i^{(1)}, p^{(1)})$. This fact and the form of the right hand side of the boundary condition (16) suggest that the solution for $(u_i^{(1)}, p^{(1)})$ should take the form

$$u_i^{(1)} = \sum_{m \neq 0} A_m W_i(x_2, m) e^{imx_1}, \quad p^{(1)} = \sum_{m \neq 0} A_m P(x_2, m) e^{imx_1}, \quad (19)$$

where A_m ($m = \pm 1, \pm 2, \dots$) are constants to be determined and from now on all the Fourier expansions are summed from $-\infty$ to ∞ with zero excluded. The functions $W_i(x_2, m)$ and $P(x_2, m)$ are determined by solving the linear eigenvalue problem mentioned above and their expressions can be found in Ogden & Fu (1996). We have

$$W_1(x_2, m) = \xi_a s_a e^{s_a |m| x_2}, \quad W_2(x_2, m) = -i(m/|m|) \xi_a e^{s_a |k| x_2},$$

$$P(x_2, m) = -im\xi_a F(s_a) e^{s_a|m|x_2} \quad (20)$$

where

$$\xi_1 = \frac{s_2^2 + 1}{s_2^2 - s_1^2}, \quad \xi_2 = \frac{s_1^2 + 1}{s_1^2 - s_2^2},$$

and s_1, s_2 are the roots of

$$s^2 = \gamma^{-1} \left(\beta \pm \sqrt{\beta^2 - \alpha\gamma} \right)$$

with positive real parts, and

$$\alpha = \mathcal{A}_{01212}^1, \quad \gamma = \mathcal{A}_{02121}^1, \quad 2\beta = \mathcal{A}_{01111}^1 + \mathcal{A}_{02222}^1 - 2\mathcal{A}_{01122}^1 - 2\mathcal{A}_{01221}^1,$$

$$F(x) = \gamma x^3 + (v^2 + \mathcal{A}_{01122}^1 + \mathcal{A}_{02112}^1 - \mathcal{A}_{01111}^1)x.$$

In (20) we have employed a modified summation convention whereby a suffix appearing more than once is summed from 1 to 2. This modified summation convention will also be observed in all subsequent analysis.

We have defined W_i and P such that they satisfy $W_i(x_2, -m) = W_i^*(x_2, m)$, $P(x_2, -m) = P^*(x_2, m)$, where a superscript “*” signifies complex conjugation. The reality of $u_i^{(1)}$ and $p^{(1)}$ then implies that $A_{-m} = A_m^*$.

The amplitude equations for A_k ($k = 1, 2, \dots$) are best derived with the aid of the virtual work method (Fu 1995, Fu & Devenish 1996). We consider a contour integral I defined by

$$I = \frac{1}{2\pi} \oint_{\partial D} \chi_{ij} n_j \hat{u}_i dS, \quad (21)$$

where n_j is the outward normal to the path, ∂D is the boundary of the rectangular region D ($0 \leq x_1 \leq 2\pi$, $-h \leq x_2 \leq 0$) where h is an arbitrary positive constant, and \hat{u}_i is a linear solution given by

$$\hat{u}_i = W_i(x_2, -k) e^{-ikx_1}, \quad (k > 0) \quad (22)$$

With the use of (16), we find that

$$\lim_{h \rightarrow \infty} I = -\frac{i}{2} \epsilon^2 \bar{\pi}_{11} (\xi_1 s_1 + \xi_2 s_2) \delta_{1k} + O(\epsilon^3). \quad (23)$$

On the other hand, the contour integral I in (21) can also be evaluated by applying the divergence theorem. This yields

$$I = \frac{1}{2\pi} \int_D \chi_{ij} \hat{u}_{i,j} dx_1 dx_2, \quad (24)$$

where use has been made of (11a). The further reduction of (24) is similar to that carried out in Fu & Devenish (1995). We substitute the expressions (16) and (17) into (13) and then the resulting expansion into (24). After the limit $h \rightarrow \infty$ is taken, the $O(\epsilon)$ term can be shown to vanish and we obtain

$$\lim_{h \rightarrow \infty} I = \frac{\epsilon^2}{2\pi} \int_{-\infty}^0 dx_2 \int_0^{2\pi} \left\{ \left(\lambda_1 \mathcal{A}_{jilk}^{1'} u_{k,l}^{(1)} + \bar{p}_1 u_{j,i}^{(1)} + \frac{1}{2} \mathcal{A}_{jilknm}^2 u_{k,l}^{(1)} u_{m,n}^{(1)} \right) \right.$$

$$-\bar{p}_0 u_{j,k}^{(1)} u_{k,i}^{(1)} + p^{(1)} u_{j,i}^{(1)} \hat{u}_{i,j} + \frac{1}{2} \hat{p} u_{i,j}^{(1)} u_{j,i}^{(1)} \Big\} dx_1 + O(\epsilon^3), \quad (25)$$

where the prime on \mathcal{A}_{jilk}^1 signifies differentiation with respect to λ and both $\mathcal{A}_{jilk}^{1'}$ and \mathcal{A}_{jilknm}^2 are evaluated at $\lambda = \lambda_0$, $\bar{p} = \bar{p}_0$.

On equating the coefficients of ϵ^2 on the right hand sides of (23) and (25), we obtain

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^0 dx_2 \int_0^{2\pi} \left\{ \left(\lambda_1 \mathcal{A}_{jilk}^{1'} u_{k,l}^{(1)} + \bar{p}_1 u_{j,i}^{(1)} + \frac{1}{2} \mathcal{A}_{jilknm}^2 u_{k,l}^{(1)} u_{m,n}^{(1)} \right. \right. \\ & \quad \left. \left. - \bar{p}_0 u_{j,k}^{(1)} u_{k,i}^{(1)} + p^{(1)} u_{j,i}^{(1)} \right) \hat{u}_{i,j} + \frac{1}{2} \hat{p} u_{i,j}^{(1)} u_{j,i}^{(1)} \right\} dx_1 \\ & = -\frac{i}{2} \bar{\pi}_{11} (\xi_1 s_1 + \xi_2 s_2) \delta_{1k}. \end{aligned} \quad (26)$$

To facilitate further analysis, we write

$$u_{m,n}^{(1)} = \sum_{p \neq 0} A_p \Gamma_{mn}(x_2, k) e^{ikx_1}, \quad (27)$$

where Γ_{mn} are calculated from (19). On substituting (27) into (26), we obtain, after some manipulations,

$$c_1 \lambda_1 |k| A_k + i |k| \sum_{k' \neq 0} \kappa(k, k') A_{k'} A_{k-k'} = -\frac{i}{2} \bar{\pi}_{11} (\xi_1 s_1 + \xi_2 s_2) \delta_{1k}, \quad (28)$$

where

$$\begin{aligned} c_1 &= -\frac{\xi_a \xi_b}{s_a + s_b} \left\{ \mathcal{A}_{nmqp}^{1'} \Gamma(p, q, k, a) \Gamma(m, n, -k, b) + \frac{\bar{p}_1}{\lambda_1} \Gamma(p, q, k, a) \Gamma(q, p, -k, b) \right\}, \\ \kappa(k, k') &= \frac{\xi_a \xi_b \xi_c |k'| |k - k'|}{s_a |k| + s_b |k'| + s_c |k - k'|} \left\{ \frac{1}{2} \mathcal{A}_{qpnmrs}^2 \Gamma(p, q, -k, a) \Gamma(r, s, k', b) \Gamma(m, n, k - k', c) \right. \\ & \quad \left. + \frac{k'}{|k'|} F(s_b) \Gamma(n, m, -k, a) \Gamma(m, n, k - k', c) - \frac{k}{2|k|} F(s_a) \Gamma(n, m, k', b) \Gamma(m, n, k - k', c) \right\}. \end{aligned}$$

3 EVOLUTION OF SURFACE ELEVATION

We now solve the amplitude equations (28) for the case when the composing material is neo-Hookean for which the strain energy function is given by $W = (1/2)(\text{tr} \mathbf{F} \mathbf{F}^T - 3)$ (after scaled by the shear modulus μ). We have

$$\boldsymbol{\pi} = \mathbf{F} - p \mathbf{F}^{-T}. \quad (29)$$

Since $\bar{\mathbf{F}} = \text{diag}(\lambda, 1/\lambda)$, we have $\bar{\boldsymbol{\pi}} = \text{diag}(\lambda - \bar{p}/\lambda, 1/\lambda - \bar{p}\lambda)$. The stress free boundary condition yields $\bar{p} = 1/\lambda^2$ and hence $\bar{\pi}_{11} = \lambda - \lambda^{-3}$. From (14) the first order elastic moduli are given by

$$\mathcal{A}_{jilk}^1 = \delta_{ik} \bar{B}_{jl}, \quad (30)$$

and all second order order elastic moduli are zero. Equation (30) may also be written as

$$\mathcal{A}_{jilk} = \delta_{ik}(\delta_{j1}\delta_{l1}\lambda^2 + \delta_{j2}\delta_{l2}\lambda^{-2}), \quad (31)$$

from which we obtain

$$\mathcal{A}'_{jilk} = 2\delta_{ik}(\delta_{j1}\delta_{l1}\lambda - \delta_{j2}\delta_{l2}\lambda^{-3}). \quad (32)$$

The other constants which are required in our calculations are

$$\begin{aligned} \alpha &= \lambda^2, & \gamma &= \lambda^{-2}, & 2\beta &= \lambda^2 + \lambda^{-2}, \\ s_1 &= 1, & s_2 &= \lambda^2, & F(s_1) &= \lambda^{-2} - \lambda^2, & F(s_2) &= 0. \end{aligned}$$

A linear stability analysis for a perfect neo-Hookean half-space shows (see e.g. Dowaiikh & Ogden 1990) that the half-space is stable if $0 < \lambda \leq \lambda_0$ where $\lambda_0 \approx 0.5437$. On substituting this value into the expression for c_1 , we obtain $c_1 \approx 10.940708$ and the right hand side of (28) becomes $-1.543689i\delta_{1k}$. The expression for the kernel $\kappa(k, k')$ can be evaluated with the aid of Mathematica and is given in an appendix at the end of this paper.

The amplitude equations clearly admit a solution given by $A_k = iB_k$ where B_k ($k = 1, 2, \dots$) are pure real. The condition $A_{-k} = A_k^*$ gives $B_{-k} = -B_k$ and in terms of B_k the amplitude equations (28) become

$$c_1\lambda_1 k B_k - k \sum_{k'=1}^{\infty} [\kappa(k, k')B_{k'}B_{k-k'} - \kappa(k, -k')B_{k'}B_{k+k'}] = -1.543689\delta_{1k}, \quad (33)$$

where $k = 1, 2, \dots$. We first truncate the summation at $k' = 2$ and assume that only B_1 and B_2 are non-zero. Setting $k = 2$ in (33) we obtain

$$B_2 = \kappa(2, 1)B_1^2/c_1, \quad (34)$$

and substituting this expression into the equation obtained by setting $k = 1$ in (33), we obtain

$$B_1^3 + k_1 B_1 + k_2 = 0, \quad \text{where } k_1 = -21.728855\lambda_1^2, \quad k_2 = -3.0658523\lambda_1. \quad (35)$$

The three roots of (35) are given by

$$B_1^{(1)} = \Phi \cos \hat{\phi}, \quad B_1^{(2)} = \Phi \cos(\hat{\phi} + 2\pi/3), \quad B_1^{(3)} = \Phi \cos(\hat{\phi} + 4\pi/3), \quad (36)$$

where

$$\Phi = 2\sqrt{-k_1/3}, \quad \cos 3\hat{\phi} = -\frac{k_2}{2}\left(-\frac{3}{k_1}\right)^{3/2} \stackrel{\text{def.}}{=} \Delta.$$

The two roots $B_1^{(2)}$ and $B_1^{(3)}$ are real only if $|\Delta| < 1$, that is if $|\lambda_1| > \lambda_{cr}$ where

$$\lambda_{cr} \approx 0.28042965.$$

The three solutions of (35) are shown in Fig.1(a) where to facilitate comparison with standard bifurcation diagrams in the literature we have shown $1/\lambda$ again $-B_1$ (so that increasing

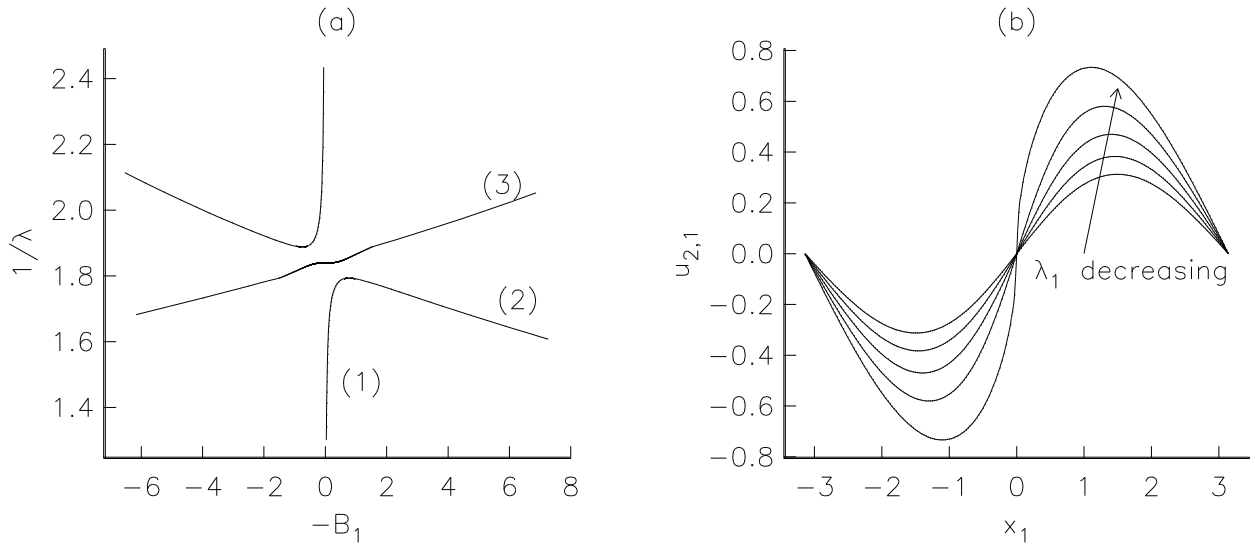


Figure 1: (a) Load-deformation curve in a two-mode approximation; (b) profiles of $u_{2,1}$ for $\lambda_1 = 0.90, 0.74, 0.61, 0.50, 0.40$.

$1/\lambda$ corresponds to increasing compression), the λ being calculated from $\lambda = \lambda_0 + \epsilon\lambda_1$ with $\epsilon = 0.05$. The superscripts (1), (2), (3) in (36) are also marked on the corresponding solution branches in Fig.1(a). The unmarked branches in Fig.1(a) can be obtained from the marked branches by using the fact that if (B_1, λ_1) is a solution of (35), then so is $(-B_1, -\lambda_1)$.

We note that the limit $\lambda_1 \rightarrow \infty$ corresponds to tending to the original stress free state. The branch of solution in Fig.1 that tends to zero as $1/\lambda$ decreases corresponds to $(36)_3$. The solution tends to zero according to $B_1 \rightarrow -0.1410959/\lambda_1$ as $\lambda_1 \rightarrow \infty$ (obtained by balancing the second and third terms in the cubic equation in (35)). This limit corresponds to a surface elevation of order $O(\epsilon B_1) = O(\epsilon/\lambda_1) = O(\epsilon^2)$ which is the order of the initial surface elevation. Thus this branch is the branch relevant to our physical problem. Fig. 1 shows that in the two mode approximation as the compression is increased gradually, the load-deformation curve has a turning point which is typical for structures sensitive to imperfections (see, e.g., Hutchinson and Koiter 1970). We define this turning point as the bifurcation point for the imperfect half-space.

We now investigate how this two-mode result is modified by the inclusion of higher modes. In our numerical calculations, we start from a suitably large λ_1 and decrease it gradually towards zero. For each λ_1 , we solve the amplitude equations (33), truncated at a finite number, with the aid of Nag library subroutine C05NBF. After solution of N equations for B_1, B_2, \dots, B_N is found, we solve $N + 1$ equations with the initial guess for the first N unknowns taken as the solution in the previous calculation and B_{N+1} set to zero. The progression starts with $N = 2$, with the solution given by (36c), and stops when a convergence criterion is satisfied. The criterion used in our calculations is that the increment of the sum of all the non-zero Fourier amplitudes is less than 10^{-15} as the truncation number is increased from N to $N + 1$.

Fig.1(b) shows the profiles of $u_{2,1}$ as λ_1 is decreased. We find that no solution exists for

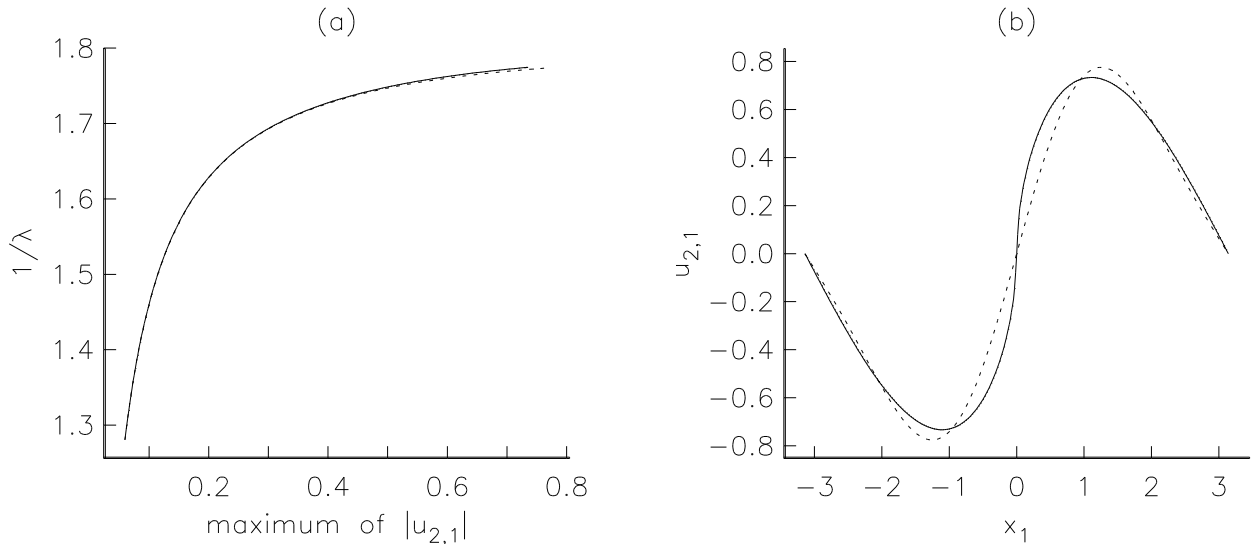


Figure 2: Comparison of results from a 2-mode (dotted line) approximation with results from a 156-mode (solid line) approximation.

$\lambda_1 < 0.4027$ which gives $\lambda > 1.7742$ with $\epsilon = 0.05$, which may be compared with $\lambda_1 < 0.2803$, $\lambda > 1.7937$ obtained from a 2-mode approximation. As λ_1 approaches 0.40270, a discontinuity seems to be developing in the profile of $u_{2,1}$. This is confirmed by the fact that as this limit is approached an increasingly large truncation number has to be used to satisfy our convergence criterion.

In Fig.2(a, b) we compare results from the 2-mode approximation with those from a 159-mode approximation. The λ is again calculated from $\lambda = \lambda_0 + \epsilon\lambda_1$ with ϵ taken to be 0.05. Fig.2(b) compares the profiles of $u_{2,1}$ when $\lambda_1 = 0.4027$. We see that although the inclusion of higher modes has a negligible effect on the amplitude of $u_{2,1}$, it has a significant steepening effect on the profile of $u_{2,1}$.

We have also tried to use branch (2) as a starting solution to find a non-trivial solution for each fixed λ_1 . However we find that the solution always converges to the trivial solution as the truncation number is increased. It is still not clear how the load amplitude curve shown in Fig.2(a) can be extended.

4 CONCLUSIONS

In this paper we have considered the stability of an imperfect neo-Hookean half-space which is subjected to a uni-axial compression. It is found that a solution exists only if the principal stretch is greater than the critical value $\lambda_0 + 0.4027\epsilon$ where λ_0 is the buckling principal stretch predicted by the linear theory for a perfect neo-Hookean half-space and ϵ^2 characterizes the amplitude of the initial modal imperfection. This critical value, which corresponds to a lower compression than λ_0 , can be defined as the buckling principal stretch for the imperfect half-space. It would be of interest to see how the load-amplitude curve shown in Fig.2(a) is extended. This is under further investigation and relevant results will appear elsewhere.

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APPENDIX: THE KERNEL $\kappa(K, K')$ FOR A NEO-HOOKEAN HALF-SPACE UNDER UNI-AXIAL COMPRESSION

For a neo-Hookean material which is subjected to a uni-axial compression, the kernel in (28) is given by

$$\kappa(k, k') = \frac{1 + \lambda^4}{2(1 - \lambda^4)^2} \tilde{\kappa}(k, k'),$$

where

$$\begin{aligned} \tilde{\kappa}(k, k') = & \frac{2(1 + \lambda^4)^2}{\lambda^2(k + |k'| + |k - k'|)} [3k'(k - k') + |k - k'| (2k' - |k'|)] \\ & - \frac{4(1 + \lambda^4)}{\lambda^2 k + |k'| + |k - k'|} [(\lambda^2 + \lambda^{-2})k'|k - k'| + 2k'(k - k')] \\ & + \frac{16\lambda^2 k'}{\lambda^2 k + |k'| + \lambda^2 |k - k'|} (|k - k'| + k - k') \\ & + \frac{2(1 + \lambda^4)}{k + |k'| + \lambda^2 |k - k'|} [(\lambda^2 + \lambda^{-2})|k - k'| (|k'| - 2k') - 6k'(k - k')] \\ & + \frac{2(1 + \lambda^4)}{k + \lambda^2 |k'| + |k - k'|} [(\lambda^2 + \lambda^{-2})|k'| |k - k'| - 2k'(k - k')] \\ & + \frac{8\lambda^2}{k + \lambda^2 |k'| + \lambda^2 |k - k'|} [k'(k - k') - |k'| |k - k'|]. \end{aligned}$$