Black hole inconsistencies?

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Note to the reader
This work is a non-specialist's attempt to make sense of the current controversy surrounding the information loss paradox in relation to gravitational collapse. Its main purpose is to elicit constructive criticism and helpful comments from experts in the field, while also drawing the attention of general readers to inconsistencies, either real or perceived. This work forms part of a personal journey through general relativity and related fields, and therefore in parts it is written in a rather informal style.

Abstract
For large enough astrophysical objects, internal pressures are insufficient to prevent gravitational collapse to sizes smaller than a critical radius where the magnitude of escape velocity is equal to the speed of light. In coordinates natural for distant observers the collapse time diverges as the critical radius is approached. On the other hand quantum evaporation as witnessed by stationary observers is traditionally modelled to take place in finite coordinate time.

In this work the coordinate invariant temporal order of non-spacelike separated events in the vacuum exterior manifold and the behaviour of inbound null trajectories are examined. Based purely on descriptions of conformal diagrams and two well-known sets of coordinates, a major inconsistency in the traditional model of an evaporating black hole is exposed.

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1. Introduction

Coordinate transformations can be visualised using the analogy of a fairground mirror. You may look into the mirror and see a distorted image of yourself. But that image is a representation of you in your current physical state. You do not, for example, see an image of you headless carrying your head under your right arm. That would be a different physical state. So we may understand coordinates themselves as a framework in which we view physical systems and their behaviour. However, it would not be correct to suggest that transforming the coordinates would change the physics of the process, only how it looks. What we need to do is to pick out key features within a dynamic process and the relationships between them that are unaffected by the transformation. It is these features that are physically significant and we should hold on to the fact that non-trivial coordinate transformations do not change the physics. This is one of the guiding principles of general relativity known as coordinate independence, or covariance.

What we have said so far may make this topic seem easy–it is anything but. The situation is relatively straightforward while we are dealing with a subset of transformations known as diffeomorphisms. A diffeomorphism is essentially a transformation that is continuous and at least once differentiable. However, when it comes to the possibility that a large enough mass can gravitate to within a one-way event horizon, situations arise that are conceptually more difficult. In the case of spherically symmetric gravitational collapse with negligible angular momentum and electric charge, the natural choice of coordinates for observers remaining at a distance is the Schwarzschild system summarized by the line element (metric) given by

$$ds^2 = \left(1 - \frac{r_c(t_S)}{r}\right) dt_S^2 - \left(1 - \frac{r_c(t_S)}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 . \tag{1}$$

Here the critical radius, $r_c(t_S) = 2M(t_S)G/c^2$, for total mass, *M*, is treated as time dependent to allow for accretion of more mass post collapse and shrinkage via quantum evaporation. For brevity we have introduced the unit 2-sphere defined by $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ for the Eulerian angles, θ and ϕ. In this metric the coordinates are singular at $r = r_c$ and $r = 0$. By transforming the coordinates it is possible to remove the singularity at the boundary ($r = r_c$), but not at the origin ($r = 0$). There the Riemann tensor dependent Kretchmann-invariant $R_{abcd}R^{abcd} \to \infty$, therefore this is a real (physical) curvature singularity. However, transformations removing the singularity at $r = r_c$ must themselves be singular at this point. Therefore they are not diffeomorphisms. By regularizing the coordinates at the critical radius one wonders whether, by application of a singular transformation, we are introducing mathematical artefacts into the description that may lead us off course from true understanding. Prior to the revelation and therefore real possibility that maximally collapsed objects may evaporate (Hawking, 1974), the transforming away of the coordinate singularity was not considered a problem. It became accepted that test particles could fall through the horizon and inevitably encounter the central singularity all within a finite proper time. Only when it was realised that these objects could evaporate that controversy and debate was fired up again. This was in no small part due to the realisation that the existence of an event horizon and central singularity might mean the permanent loss of information from the universe, which threatened the cherished law of unitary evolution.

Because the classical black hole was fully characterised by at most three parameters: mass, angular momentum and electric charge, it was realised that information about anything that fell in would be lost to the outside universe. This is related to the now famous *no hair* theorems (Misner, Thorne and Wheeler, 1973). As long as black holes had unlimited life expectancy this was not a problem because although information about material that had crossed the horizon was lost to the outside, it was not lost to physical reality as a whole. However, with the realisation that black holes may evaporate, information loss became problematic because the radiation left was thought to be totally thermal. The physics community became divided into two camps: those willing to accept information loss and those who would not. My own position is in accord with the latter group who it seems are in a growing minority.

In the beginning it was thought that Hawking radiation required an event horizon, however work done in later years (e.g. Unruh, 1976; DeWitt, 1979) suggested that this might not be the case and consequently the radiation emitted might not be totally thermal. Inspection of the metric in equation (1) suggests that the surface of a collapsing star would always remain above the horizon due to the diverging time dilation there. Coupled with simultaneous evaporation, the finite life expectancy would mean that the classical black hole never forms because it is overtaken by evaporation, a number of distinct approaches have been suggested to support this idea (e.g. Barcelo *et al*, 2006; Vachaspati, Stojkovic D and Krauss LM, 2007; Hawking, 2014; Mersini-Houghton, 2014). Moreover Hawking (2014) suggests the possibility that unitary evolution can be preserved without violating the CPT theorem. DeWitt (1979, p695-8) provides a fairly accessible description of the process, which is briefly summarised as follows.

Everywhere outside the horizon we may define a global timelike Killing vector, $\partial/\partial t_S$, with respect to which a vacuum state may also be defined. In particular, at large distances where the manifold is flat we expect the renormalized energy tensor, $\langle T^{ab} \rangle$, to vanish. But because the manifold is locally Minkowskian, a small enough freefalling observer would regard the vacuum close to the horizon as indistinguishable from the flat regions far away. Therefore $\langle T^{ab} \rangle$ vanishes there also. However, the story for an observer holding station just above the horizon at $r = r_c(1+\varepsilon)$ is very different. This observer may be considered equivalent to one undergoing acceleration, *a*, in Minkowski space-time, and because of this the renormalized $\langle T^{ab} \rangle$ no longer vanishes (Unruh, 1976; DeWitt, 1979, p690-5). This is known as the Rindler vacuum where, in Planck units ($G = c = \hbar = k = 1$), the temperature associated with the non-zero $\langle T^{ab} \rangle$ is $T = a/(2\pi)$. For our stationary observer the acceleration is $a = 1/\left(2r_c\sqrt{\varepsilon}\right)$, so the temperature it sees is $T = 1/\left(4\pi r_c\sqrt{\varepsilon}\right)$. However the resulting outgoing radiation undergoes a redshift of $\sqrt{\varepsilon}$ as it climbs out of the gravity well, therefore the temperature seen by a distant observer is $T = 1/(4\pi r_c)$, see equation (5) below.

The above description admittedly glosses over quite a lot of detail. However, we should bear in mind that in this work we are only considering crude toy models that break down near the final evaporation point. There are two important reasons for this:

1. We are only considering Schwarzschild space-time, which is a vacuum solution outside of the spherical mass, *M*. Close to the evaporation point it is expected that Hawking radiation density becomes comparable with the evaporating object. This is more accurately described by the Vaidya space-time.

2. Also close to the evaporation point the physics enters the Planckian regime where it is entirely possible that even the Vaidya metric breaks down. Here the physical description will require an empirically backed, universally agreed theory of quantum gravity.

However, aside from this, there are still contentious issues surrounding the evaporation of a Schwarzschild black hole and it is these inconsistencies that we address in this work. I have already expressed my concerns about this elsewhere (Austin, 2020) but not in the same detail as here. Here we work in the language of coordinates to maintain clarity for general readers, and therefore we avoid more powerful techniques using for example tetrad formalisms. A disadvantage of our approach however is that it is more error prone.

Equation (1) splits space-time into two regions; the exterior and interior labelled I and II respectively, which are delimited by the event horizon. Further we define $\mathcal{M}$ to be the manifold outside of the critical radius and $\mathcal{M}'$ is the analytically extended manifold with only the central singularity removed. Moreover we assume that $\mathcal{M}$ is an open set that is everywhere time orientable. That is to say, we can define a global metric on $\mathcal{M}$ of the form

$$ds^2 = dt^2 - g_{ij}dx^i dx^j \qquad (i, j = 1,2,3) \tag{2}$$

where t is a global time. Equation (1) satisfies this form therefore we proceed on the assumption that Schwarzschild time, t_S, can act as a global time parameter everywhere in $\mathcal{M}$.

However, this still leaves us with the coordinate singularity at the critical radius. This is why t_S cannot be considered global in $\mathcal{M}'$. In the past this singularity has been regularised so that the boundary appears as a smooth traversable event horizon. There are many appropriate coordinate systems with this property, but in this work we choose the earliest discovered, which are Painlevé-Gullstrand (P-G) coordinates given by

$$ds^2 = \left(1 - \frac{r_c(t_{PG})}{r}\right)dt_{PG}^2 - 2\sqrt{\frac{r_c(t_{PG})}{r}}dt_{PG}dr - dr^2 - r^2 d\Omega^2 . \tag{3}$$

This metric in particular has the advantage that it reduces to Minkowski space-time at large distances and in the zero-mass limit. Moreover it is a natural choice for inward freefalling observers. However, it is noted that in equation (3) the time variable is not orthogonal to the spatial coordinates and therefore does not fit the form of equation (2).

In the next section conformal diagrams are briefly introduced and then discussed in relation to gravitational collapse. Inconsistencies with respect to the global time parameter, t_S, are highlighted in the traditional representation (figure 3) of gravitational collapse and subsequent evaporation. In section 3 we compare two distinct evaporation models (Aste and Trautmann, 2005; Kassner, 2019). It is shown that Kassner's model, while consistent in the context of P-G coordinates, exhibits inconsistencies with regard to the global time variable, t_S, which are reminiscent of those seen in figure 3.

2. Conformal diagrams

These are compact schematic illustrations used to represent asymptotically flat space-time regions using a chosen conformal factor that compresses them into finite sized figures. Roger Penrose and Brandon Carter first employed them in the 1960s as an aid to the analysis of gravitational collapse, singularities and asymptotic behaviour. As a consequence they are variously known as Penrose diagrams, Penrose–Carter diagrams or Carter–Penrose diagrams. A crucially important feature, because the null-cone structure is a key to determining causal relations in the manifold, is that $45°$ diagonals cross in the diagram if and only if their corresponding light rays cross.

As an example Figure 1 is a representation of Minkowski (flat) space-time. In Figure 1a we see a full diamond with the upper and lower vertices, $i^{\pm}$, representing past and future timelike infinities with time increasing in an upward direction. Similarly the left and right vertices are spacelike infinity, i^0, in either direction of a chosen spatial coordinate. The diagonal sides, $\mathscr{I}^{\pm}$, represent the future (past) null infinity for the upper (lower) half of the diagram. Points in the diagram are full spacelike surfaces in the other two spatial coordinates. The half-diamond shown in figure 1b is often (but not always) used where a non-negative radial axis forms part of the coordinate system. Examples include spherical coordinates in Minkowski space-time, or Schwarzschild (Boyer-Lindquist) coordinates where a massive non-rotating (rotating) body is centred at the origin.

In figure 1 we see hyperbolae that converge on the vertices. These are constant time and distance coordinates, (t,x), where, in this case, we have chosen $\tan(u \pm v) = x \pm t$ which transform space-time coordinates (t,x) to Penrose coordinates (v,u). However, in general there are many ways to transform a manifold in order for it to fit into a conformal diamond with null trajectories on the $45°$ diagonals.

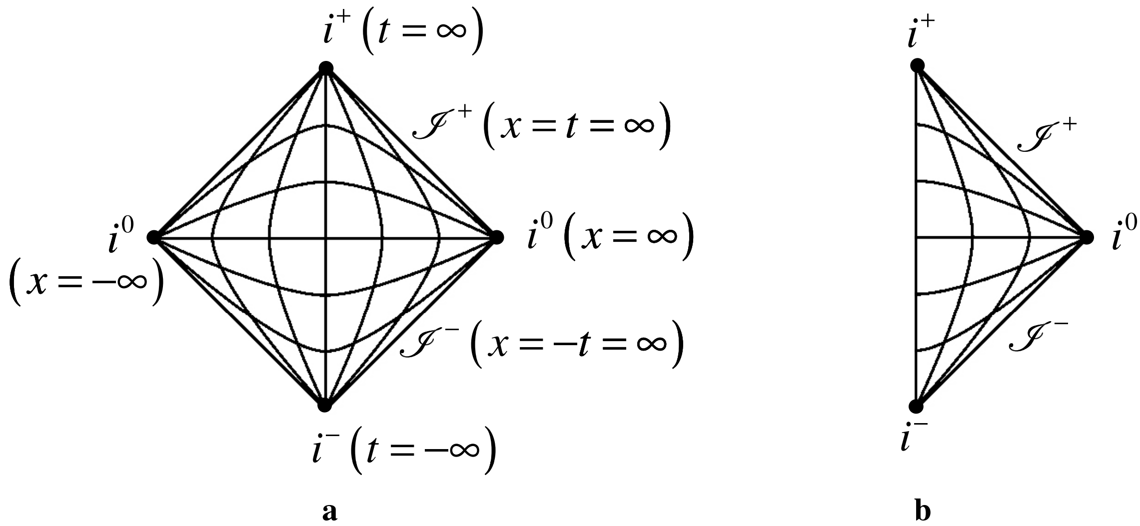

Figure 1: a A conformal diagram for Minkowski space-time with time on the vertical axis and distance in either direction in the horizontal. The infinite timelike past and future are concentrated at the points i^- and i^+ respectively. The diagonal boundaries, $\mathscr{I}^{\pm}$, are correspondingly null past and future, and spacelike infinity is at i^0. In arbitrary Gaussian coordinates each point represents a 2D spacelike surface. **b** In cases where radial coordinates are employed a half-diamond may be used where the origin of the coordinate system is the left hand edge. Individual points here represent closed 2D spacelike surfaces with topology, S^2.

Application to black holes

Conformal diagrams are readily applied to classical black holes as they were considered prior to Hawking's (1974) proposed evaporation mechanism. Before this, black holes were thought of as the ultimate stable state from which no signal could escape, the only measurable parameters being mass, charge and angular momentum. This is connected to *no hair* theorems (Misner, Thorne and Wheeler, 1973) where information is effectively lost but not destroyed.

In figure 2 we have a conformal diagram of a normal exterior space-time (region I) with an extra interior region (II) *bolted on*. It is possible to enter region II from region I following a timelike trajectory across a one-way event horizon (*EH*). The horizontal double line at the top of the diagram represents a spacelike singularity, an encounter with which is inevitable for any observer in region II having crossed event horizon. This diagram represents a non-rotating, electrically neutral, classical Schwarzschild black hole with infinite life expectancy. In the diagram, an indicator of the unlimited life of a classical black hole is the event horizon (*EH*) extending all the way to the infinite timelike future, i^+.

As long as these objects are considered ultimately stable then there is no problem around the issue of information loss. In the inset of figure 2, a freefalling observer B crossing the horizon will be spacelike separated from an event sufficiently far into the future on observer A's worldline. Therefore information about B still exists even though it is effectively lost. As we will see in the next subsection this is not the case when we consider the possibility that black holes may evaporate entirely. An important question is: is this really possible?

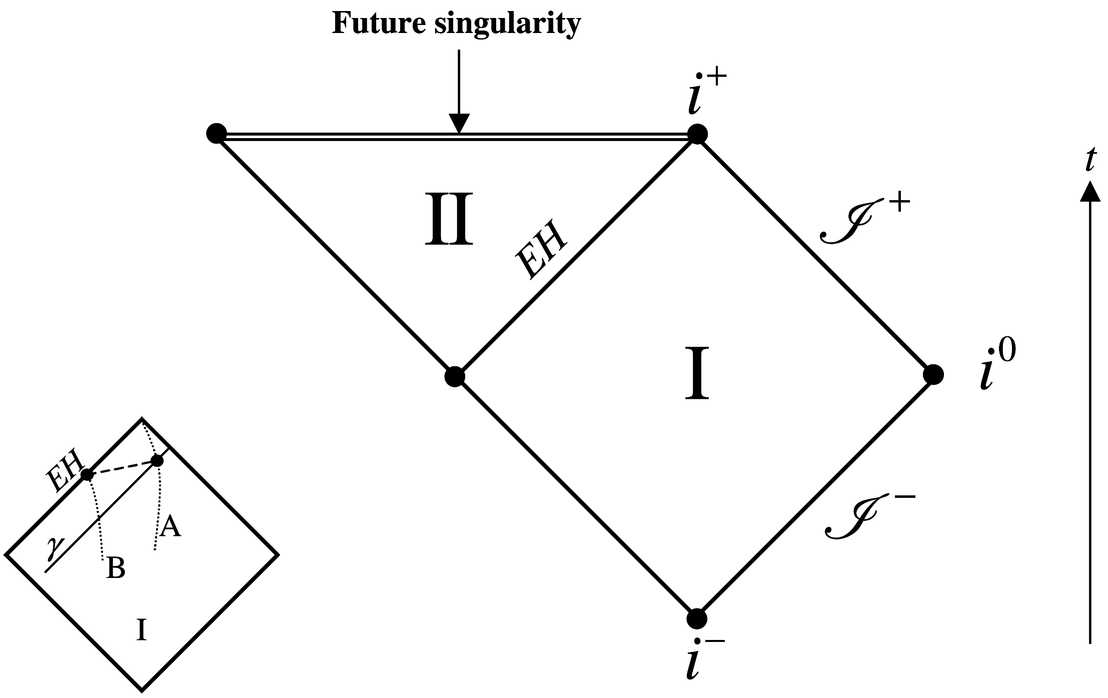

Figure 2: Conformal representation of a non-rotating, electrically neutral Schwarzschild black hole. The event horizon (*EH*) is the boundary between the interior (region II) and the rest of the universe (region I). t is a global coordinate time that increases in an upward direction in the diagram. (The inset indicates that the point where observer B crosses the horizon is spacelike separated from distant observer A at a point crossing the photon path, γ, sufficiently far into the future.)

Evaporating black holes

Although it is generally accepted that black holes or gravitationally collapsed objects evaporate via a partially understood interaction with a quantum vacuum based on the proper acceleration of shell observers, the question of the exact model describing this process is still open. However, for our purposes we do not need such details. For the present discussion we divide evaporation models into two broad categories: (i) models leading to complete evaporation and (ii) models leading to a remnant or predicting no evaporation. In the latter case one may apply conformal representations of the type seen in figure 2. This is because remnants have an unlimited life expectancy just like the non-evaporating case. We will therefore confine the discussion to the former case for the rest of this section.

The traditional conformal representation for a black hole that evaporates completely is shown in figure 3. The black hole interior is bounded by the spacelike singularity denoted by the double horizontal line, the event horizon (*EH*) and the timelike centre at $r = 0$. An observer, B, originating in the exterior region at an earlier time subsequently crosses the horizon. All information about observer B is destroyed at the singularity. Because the black hole evaporates entirely at the point where *EH* meets the singularity all information about material that has crossed *EH* in the past is irrevocably destroyed, therefore a Cauchy horizon, *CH*, is formed. The evaporation for a typical stellar sized black hole takes place in a very long ($\sim 10^{69}$ years) but finite coordinate time as seen by observer A who remains in the exterior region and crosses *CH*.

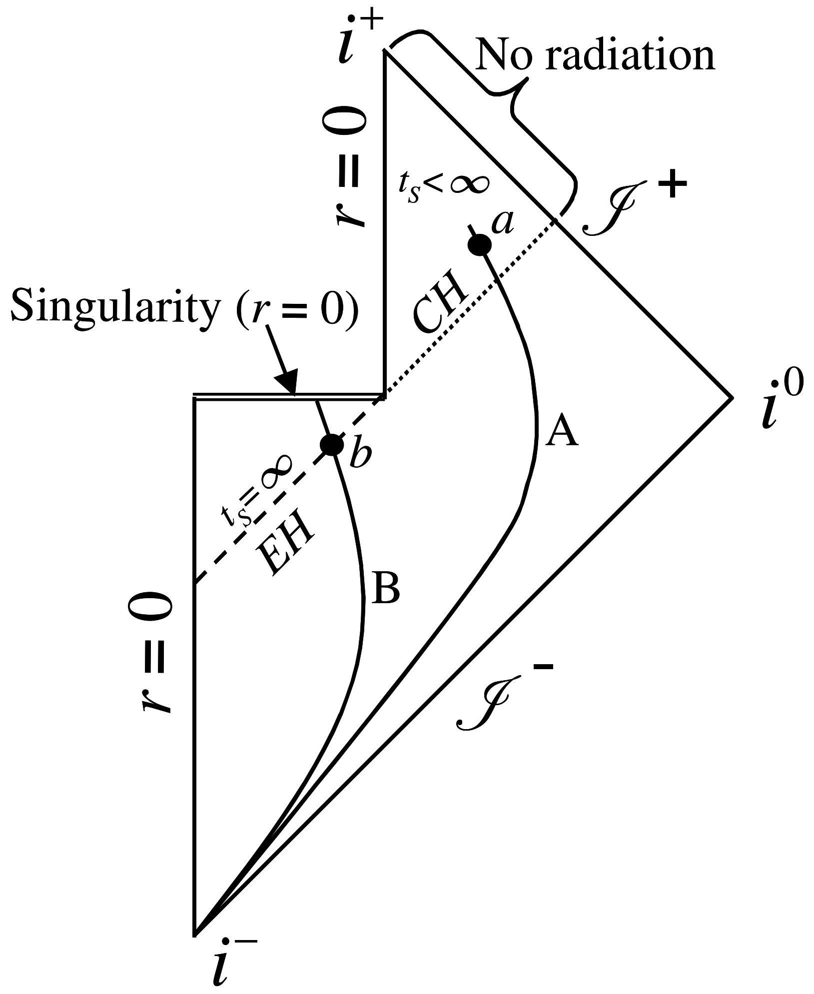

Figure 3: A traditional conformal diagram representing an evaporating black hole. Observer B enters the interior at *b* and encounters the singularity. Observer A avoids the black hole and crosses the Cauchy horizon (*CH*) to reach the later event *a*. In Schwarschild coordinates a contradiction arises due to $t_S(a) < t_S(b)$ where in the diagram these events have the opposite temporal order.

The process depicted in figure 3 appears to have one glaring contradiction, which relates to how we might label coordinate time within the diagram. Observers A and B each evolve according to their own proper time, and therefore there is no contradiction there. It is the coordinates to which we relate our calculations where problems arise, and it is appreciated that this can generate much confusion. Conformal diagrams are useful because they strip away everything but the bare essentials when it comes to descriptions of the dynamics in curved space-time, in particular where event horizons and singularities are concerned. General relativity is built on the notion of coordinate invariance (covariance), so we need to identify which features of a particular process are invariant. One such feature is the temporal ordering of non-spacelike separated events, and conformal diagrams make it very easy to identify the relationship between any such pair of events. So it should be possible to choose two sets of coordinates and there should be parity between the two relating to all the invariant features. In other words there should be no difference between the conformal diagrams associated with the respective coordinate choices. In this work we consider Schwarzschild coordinates, which are singular at *EH*, and Painlevé-Gullstrand (P-G) coordinates, which are regular at *EH*.

Consider the following two non-spacelike separated events: the point, *b*, where the worldline of B intersects *EH* and a point, *a*, on A's worldline beyond the Cauchy horizon. On *EH* at event *b* the Schwarzschild time $t_S = \infty$. Now, from event *b* we can follow a future directed null path to the evaporation point and then from there, a future directed timelike path to event *a*. This constitutes a contradiction because we have an event, *a*, to the future of an event, *b*, but with a smaller value of t_S. Now it might be argued that because Schwarzschild coordinates are badly behaved at *EH*, the pathological behaviour regarding time ordering might be expected. So we should simply change the coordinates and that seems to be the prevailing wisdom. However, the problem has not really gone away. We can see this by adjusting event *b* slightly so that it is just outside of the horizon. In this way we can remove the event horizon and its interior entirely from the manifold and we are now working solely in $\mathcal{M}$, any change in coordinates would now constitute a diffeomorphism and we would have removed the offending features. So repeating the same procedure we can still follow a non-spacelike trajectory from *b* to *a*. However, the problem remains because we can always choose event *b* to be close enough to *EH* for it to reach a value of t_S greater than that at event *a*. Therefore there is an inherent contradiction in the diagram. The only way, it seems, to resolve this situation is to use a form of coordinates where global time is finite on *EH*.

Assuming a finite coordinate time, which of course can vary along *EH* as the diagram implies, would allow inbound null geodesics to cross *EH* into the interior. The question now is: can this really happen, especially given that the evaporation point is firmly fixed at a finite point in the future? We know that the Schwarzschild solution is valid everywhere in $\mathcal{M}$, and regular coordinate transformations should not change the physics there. This solution tells us that inbound photons do not reach *EH* before the evaporation point. If this is true in Schwarzschild coordinates then it must also be true in, for example P-G coordinates. A possible modification to the way we use figure 3 that might resolve the issue is shown in figure 4. Here the diagram has not really changed only the way it is used.

Figure 4 consists of the same essential features as figure 3; the only essential difference is that now we have partitioned what we have previously considered to be the exterior region into $\mathcal{B}$ from which the interior is accessible and $\mathcal{E}$ from which it is not. The boundary between these two regions is an inbound null geodesic, $\boldsymbol{n}$, ending at the evaporation point. Thinking this way seems to have some very odd consequences, especially when we consider definitions for the regions $\mathcal{B}$ and $\mathcal{E}$. Comparing this with figure 2 we have the interior labelled as II, but region I is partitioned according to

$$\mathrm{I} = \mathcal{B} \cup \boldsymbol{n} \cup \mathcal{E}\,. \tag{4}$$

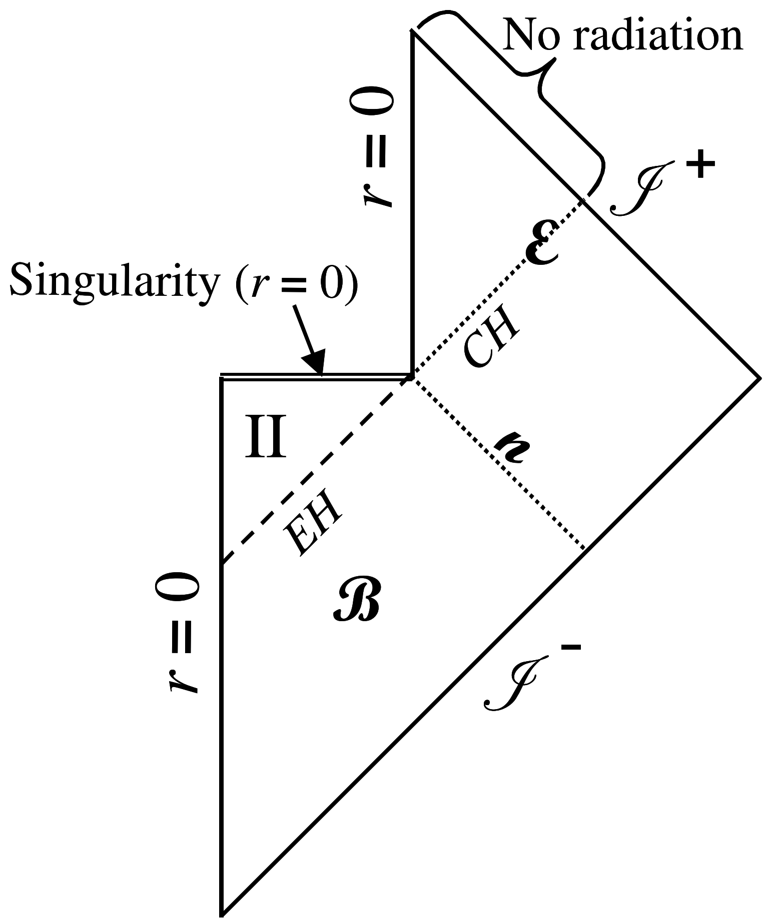

Figure 4: Conformal diagram showing region I partitioned into three sets: $\mathscr{B}$, n and $\mathcal{E}$ where n is the boundary between $\mathscr{B}$ and $\mathcal{E}$. Also $\mathcal{E}$ includes the Cauchy horizon (*CH*) and the region beyond it. In Schwarzschild coordinates the whole region, $\mathscr{B}$, is positioned at the critical radius. This therefore may be regarded as an empty set. In coordinates that are regular at the boundary $\mathscr{B}$ may be nonempty but covariance dictates that physically it is a vacuum.

In Schwarzschild coordinates recall that inbound photons do not reach the horizon. In the non-evaporating or remnant case they approach the horizon asymptotically. In the total evaporation case the vacuum solution will break down close to the evaporation end point because the energy density of the outbound Hawking flux is comparable to that of the black hole. In this regime the Vaidya solution would be more appropriate. However, this is also the point where the physics is likely to be Planckian, and is therefore beyond the bounds of current research. If we exclude *EH* and region II from the manifold we still include photons inbound from infinity all of which reach a small neighbourhood of the evaporation point. In figure 4 such photons are just outside the null boundary, n. Included in n are inbound photons that have been part of the black hole boundary from its original formation and are thus at the same critical radius as the outbound photons forming *EH*. We therefore designate the region, $\mathscr{B}$, as the set of boundary points; they are not part of the exterior at large, $\mathcal{E}$, from which all subsequent inbound matter originates. Observers in $\mathcal{E}$ can look inward and see a totally black disc as expected. This is because there are no photon sources in $\mathscr{B}$ given that it has zero 3-volume measure, at least in Schwarzschild coordinates.

If all material originates in $\mathcal{E}$ then where does this leave $\mathcal{B} \cup EH \cup \text{II}$? If it is taken that region I includes the Cauchy horizon and the region to its future then, according to equation (4) so does $\mathcal{E}$. This suggests that all material originates in $\mathcal{E} \cup \boldsymbol{n}$ and remains there always. By excluding $\mathcal{B} \cup EH \cup \text{II}$ in figure 4 from the manifold, we obtain a conformal diagram with similar topology to that shown in figure 5, where we have a body of material that undergoes full gravitational collapse and subsequent evaporation without ever forming region II or an event horizon, see e.g. (Vatchaspati, Stojkovic and Krauss, 2007). We also see that $\mathcal{M} = \mathcal{E} \subset \text{I}$ and therefore $\mathcal{M} \neq \text{I}$. The black hole boundary set, if it exists at all, should now be defined by $\{r = r_c\} = EH \cup \mathcal{B} \cup \boldsymbol{n}$.

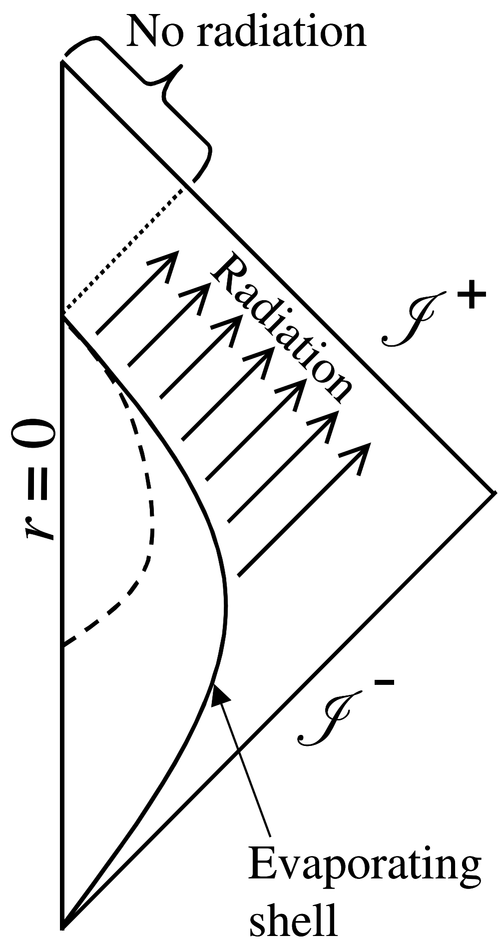

Figure 5: Conformal diagram for a collapsing shell evaporating by quasi-Hawking radiation where its non-thermal character allows information about the shell to leak away back into this universe (Vachaspati, Stojkovic and Krauss, 2007). An apparent horizon is formed, which is denoted by the dashed line. Since information is not destroyed in this process the dotted null geodesic at the top of the diagram is not a Cauchy horizon.

Because falling material hangs around above the horizon (due to gravitational time dilation) while the maximally collapsed object is evaporating, the emitted radiation is impressed with information from that material. This is not only consistent with unitary evolution but also there is a proposal by Hawking (2014) making this general idea compatible with the CPT theorem also. In that case the whole process just reduces to an information-preserving scattering problem.

3. Evaporation toy models

From semi-classical gravity it is known that a maximally collapsed object's surface is not absolutely cold, but radiates with the Bekenstein-Hawking temperature given by

$$T_{BH} = \frac{\hbar c^3}{8\pi kGM}, \tag{5}$$

where M is the mass of the object (Hawking, 1974). In Planckian units ($G = c = \hbar = k = 1$) this reduces to $T_{BH} = 1/(8\pi M) = 1/(4\pi r_c)$. In these same units the critical radius for this mass is $r_c = 2M$, and the surface area is $A = 16\pi M^2$. We can insert these facts into an equation for output power expressed as $P = A\sigma T_{BH}^4$ where in Planckian units the Stephan-Boltzmann constant is $\sigma = \pi^2/(60)$. Because the power, P, corresponds to a reduction in mass then we have $P = -dM/dt$ therefore the rate at which mass is lost is given by

$$\frac{dM}{dt} = -\frac{1}{15360\pi M^2}. \tag{6}$$

Solving this provides us with a prototype evaporation model based purely on the Bekenstein-Hawking temperature given in equation (5). By letting the evaporation time be t_0 this gives

$$M = \begin{cases} \kappa(t_0 - t)^{1/3} & t \le t_0 \\ 0 & t > t_0 \end{cases}, \qquad \kappa = \left(\frac{1}{5120\pi}\right)^{1/3}. \tag{7}$$

The question that is left open at this point is: what time variable is specified by t? In what follows we consider two toy models, one where $t = t_S$ (Aste and Trautmann, 2005) and another where $t = t_{PG}$ (Kassner, 2019). It is shown that this is not just a simple matter of coordinate change; they are distinct evaporation models with qualitatively different physical outcomes.

Aste and Trautmann: total evaporation

Considering the circumstances where $t = t_S$ and given the standard Schwarzschild metric in equation (1) we can write down an effective Lagrangian in the (r,t) plane for a freefalling test particle (observer) given by

$$L = A(r,t)\dot{t}^2 - A(r,t)^{-1}\dot{r}^2$$

where $A(r,t_S) = 1 - r_c(t_S)/r$ and the resulting partial derivatives are

$$\frac{\partial L}{\partial r} = A_r \dot{t}^2 + A^{-2} A_r \dot{r}^2$$
$$\frac{\partial L}{\partial \dot{r}} = -2A^{-1}\dot{r}$$
$$\frac{\partial L}{\partial t} = A_t \dot{t}^2 + A^{-2} A_t \dot{r}^2$$
$$\frac{\partial L}{\partial \dot{t}} = 2A\dot{t}.$$

Here $A_r \equiv \partial_r A$ and $A_t \equiv \partial_t A$. By substituting these into the corresponding Euler-Lagrange equations we arrive at

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow$$
$$-2A^{-1}\ddot{r} + 2A^{-2}A_r\dot{r}^2 + 2A^{-2}A_t\dot{t}\dot{r} = A_r\dot{t}^2 + A^{-2}A_r\dot{r}^2 \Rightarrow$$
$$-2A^{-1}\ddot{r} + A^{-2}A_r\dot{r}^2 + 2A^{-2}A_t\dot{t}\dot{r} = A_r\dot{t}^2 \Rightarrow$$
$$\ddot{r} = -\tfrac{1}{2}AA_r\dot{t}^2 + \tfrac{1}{2}A^{-1}A_r\dot{r}^2 + A^{-1}A_t\dot{t}\dot{r} \tag{8}$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} \Rightarrow$$
$$2A\ddot{t} + 2A_r\dot{r}\dot{t} + 2A_t\dot{t}^2 = A_t\dot{t}^2 + A^{-2}A_t\dot{r}^2 \Rightarrow$$
$$2A\ddot{t} + 2A_r\dot{r}\dot{t} + A_t\dot{t}^2 = A^{-2}A_t\dot{r}^2 \Rightarrow$$
$$\ddot{t} = \tfrac{1}{2}A^{-3}A_t\dot{r}^2 - A^{-1}A_r\dot{r}\dot{t} - \tfrac{1}{2}A^{-1}A_t\dot{t}^2 \tag{9}$$

thus providing two coupled second order non-linear differential equations in r and t_S. Moreover by keeping the four-velocity of the freefalling test particle constant we get

$$A(r,t)\dot{t}^2 - A(r,t)^{-1}\dot{r}^2 = 1. \tag{10}$$

Aste and Trautmann solve equations (8-10) numerically and obtain a plot in which the radius of the test particle, *r*, approaches but does not reach r_c before complete evaporation ($r_c = 0$). It is not difficult to deduce this kind of behaviour just by inspection of the standard Schwarzschild metric in equation (1). Taken as a whole the evolution of this system is entirely consistent and does not lead to any incoherent behaviour. There are no singularities and no event horizon because the collapsed object evaporates before any of these features can form. We next turn our attention to Kassner's model.

Kassner's model

Kassner (2019) adopts a similar strategy but instead focuses on the metric in P-G coordinates shown in equation (3). By removing the horizon at $r = r_c(t_{PG})$ and the interior from the manifold we have identical geometry to that is described by equation (1). Moreover the physics should be identical also, however, not only does Kassner change the coordinates to render them regular at the horizon, he also changes the evaporation model so that $t = t_{PG}$ in equation (7). This leads to qualitatively different behaviour as he freely acknowledges. It appears that Kassner is inspired by a desire for a consistent behaviour throughout $\mathcal{M}'$–a position I can sympathise with to a point. So let us summarise Kassner's analysis and see where it leads.

As with Aste and Trautmann's model we begin by writing down an effective Lagrangian based on the metric in equation (3).

$$L = A(r,t)\dot{t}^2 - 2B(r,t)\dot{t}\dot{r} - \dot{r}^2$$

where $A(r,t_{PG}) = 1 - r_c(t_{PG})/r$ and $B(r,t_{PG}) = \sqrt{r_c(t_{PG})/r}$. The required partial derivatives are listed as follows

$$\frac{\partial L}{\partial r} = A_{,r}\dot{t}^2 - 2B_r\,\dot{t}\dot{r}$$
$$\frac{\partial L}{\partial \dot{r}} = -2B\dot{t} - 2\dot{r}$$
$$\frac{\partial L}{\partial t} = A_{,t}\dot{t}^2 - 2B_t\dot{t}\dot{r}$$
$$\frac{\partial L}{\partial \dot{t}} = 2A\dot{t} - 2B\,\dot{r}$$

where again the subscripted *A*s and *B*s are the corresponding partial derivatives. Inserting these into the Euler-Lagrange equations we get

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow$$
$$-2B\ddot{t} - 2B_r\dot{r}\dot{t} - 2B_t\dot{t}^2 - 2\ddot{r} = A_r\dot{t}^2 - 2B_r\,\dot{t}\dot{r} \Rightarrow$$
$$-2B\ddot{t} - 2B_t\dot{t}^2 - 2\ddot{r} = A_r\dot{t}^2 \Rightarrow$$
$$\ddot{r} = -\tfrac{1}{2}A_r\dot{t}^2 - B\ddot{t} - B_t\dot{t}^2 \tag{11}$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} \Rightarrow$$
$$2A\ddot{t} + 2A_r\dot{r}\dot{t} + 2A_t\dot{t}^2 - 2B_t\dot{r}\dot{t} - 2B_r\dot{r}^2 - 2B\ddot{r} = A_t\dot{t}^2 - 2B_t\,\dot{t}\dot{r} \Rightarrow$$
$$2A\ddot{t} + 2A_r\dot{r}\dot{t} + 2A_t\dot{t}^2 - 2B_r\dot{r}^2 - 2B\ddot{r} = A_t\dot{t}^2 \Rightarrow$$
$$A\ddot{t} = \tfrac{1}{2}A_t\dot{t}^2 - A_r\dot{r}\dot{t} - A_t\dot{t}^2 + B_r\dot{r}^2 + B\ddot{r} \Rightarrow$$
$$A\ddot{t} = -A_r\dot{r}\dot{t} - \tfrac{1}{2}A_t\dot{t}^2 + B_r\dot{r}^2 + B\ddot{r}\,. \tag{12}$$

Again we have arrived at the two coupled second order differential equations of motion in the (r,t_{PG}) plane. For the purposes of simplification we can also write down an equation for constant four-velocity (freefall) based on the above Lagrangian, this gives

$$A\dot{t}^2 - 2B\dot{r}\dot{t} - \dot{r}^2 = 1. \tag{13}$$

Here we may also note the useful equations to aid further analysis

$$A_r = \frac{r_c}{r^2}, \quad A_t = -\frac{r_{c,t}}{r}$$

$$B_r = -\frac{1}{2r}\sqrt{\frac{r_c}{r}}, \quad B_t = \frac{r_{c,t}}{2\sqrt{rr_c}}$$

$$A + B^2 = 1, \quad \frac{2BB_r}{A_r} = -1$$

By substituting equation (12) into (11) to eliminate $\ddot{t}$ and using equation (13) to eliminate $\dot{t}$ in terms not containing A_t or B_t, we arrive at the significantly simplified form

$$\ddot{r} = -\frac{r_{c,t}}{2\sqrt{r_c r}}\dot{t}^2 - \frac{r_c}{2r^2}. \tag{14}$$

For the limiting case, $r_{c,t} \to 0$, this reduces to Newton's model as expected (Kassner, 2019).

Treating this purely in the context of P-G coordinates it is entirely consistent. But reservations begin to surface when we question the physical basis of this evaporation model. The model presented in equation (7) is based on vacuum excitations due to the necessary proper acceleration required to hold station at a fixed radius. The natural choice of coordinates for a corresponding shell observer is encapsulated in the standard Schwarzschild metric (equation (1)). Therefore it seems we should keep the same evaporation model and note that P-G coordinates (equation (3)) are a natural choice for freefalling observers that experience no acceleration. This implies no excitations and hence the temperature felt by a freefalling detector is absolute zero. So it is difficult to attach any physical basis for this evaporation model ($t = t_{PG}$), at the very least it seems that we cannot base it on DeWitt's (1979) description.

However, if we continue our analysis with this model, what can we say about different observers being released from momentary stationary points at correspondingly different radii? Further, how do these observers related to any putative evaporation process?

Figure 6 shows freefall trajectories, numerically calculated by Kassner (2019), for five observers initially falling from rest at different values of *r*. A sixth observer falling from $r_0 = r_c(1+\varepsilon)$ for $\varepsilon << 1$ has been added to illustrate the point that this model allows an observer to fall through the horizon for arbitrarily small reductions in r_c due to evaporation. Observers initially falling from rest at the same time (t_{PG}) but at larger radii do cross the horizon at later times when significant evaporation has taken place. Indeed for an observer falling from rest at a sufficiently large radius the black hole will have evaporated completely before reaching the horizon. Moreover, as Kassner comments during his closing arguments, an observer beginning a freefall at an appropriate radius and time will experience a brief repulsion during the closing stages of evaporation. This is due to $r_{c,t}$ in equation (14) (Kassner's equation (36)) being negative as a result of the critical radius decreasing with increasing time. For these reasons we might expect a similar behaviour near the evaporation point in Aste and Trautmann's model. But to pursue this discussion any further would take us too far from the main point, which revolves around the question of whether Kassner's evaporation model is consistent in a more general sense.

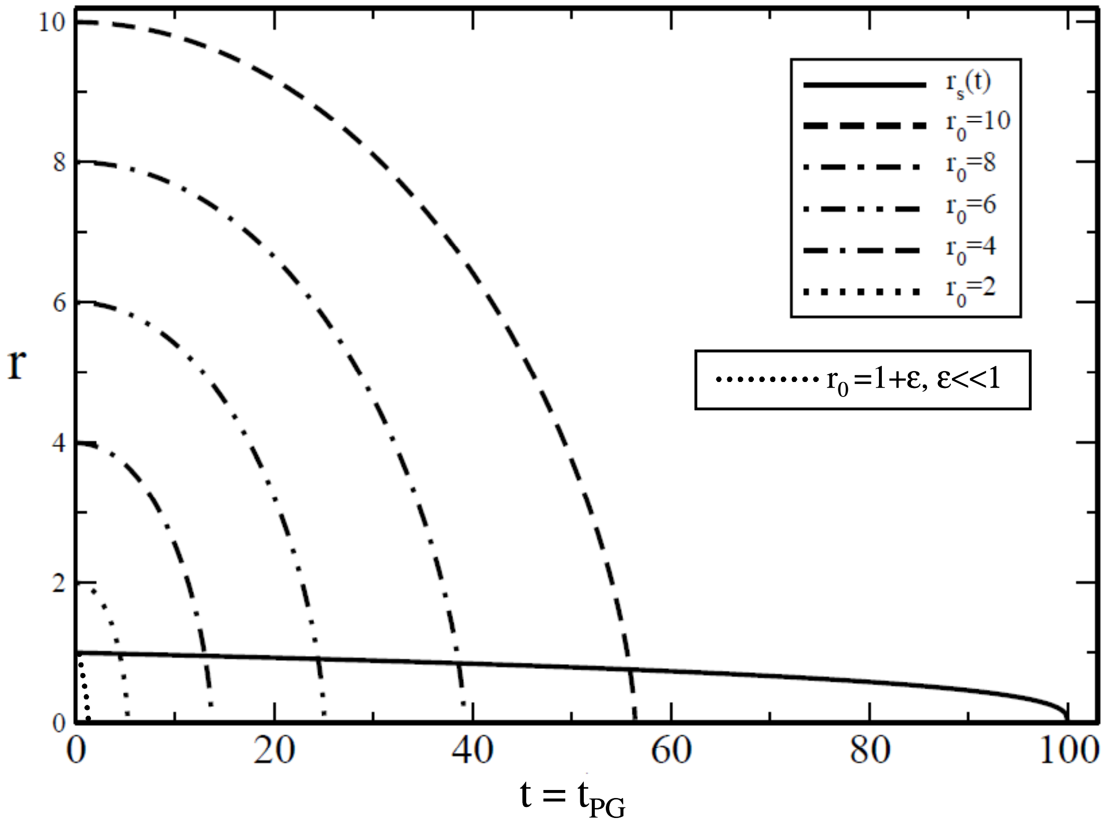

Figure 6: Trajectories of particles falling from different multiples of r_c (Kassner's r_s) towards an evaporating black hole (taken from Kassner, (2018, ArXiv version)). An extra trajectory has been added to the original for a particle falling from $r_0 = r_c(1+\varepsilon)$ for very small ε.

To explore this we may begin by considering the observer freefalling from a very small radius, $r_0 = r_c(1+\varepsilon)$, we see that at the point where the horizon is crossed there has been negligible evaporation while the observer is above the horizon. To look at this scenario in Schwarzschild coordinates we employ the transformation given by

$$t_{PG} = t_S + 2\sqrt{rr_c} - r_c \ln\left(\frac{\sqrt{r}+\sqrt{r_c}}{\sqrt{r}-\sqrt{r_c}}\right) + t_R, \qquad r_{PG} = r_S = r \tag{14}$$

where t_R is an integration constant which allows us to synchronize the various proper times at $t_{PG} = 0$ as depicted in figure 6. Note that there is no suggestion that proper time for each freefalling observer is synchronized throughout. Time for each observer can be related but only in a way that is dependent on relative movement and positioning within the manifold. Schwarzschild time is obtained from the corresponding inverse transformation

$$t_S = t_{PG} - 2\sqrt{rr_c} + r_c \ln\left(\frac{\sqrt{r}+\sqrt{r_c}}{\sqrt{r}-\sqrt{r_c}}\right) - t_R, \qquad r_S = r_{PG} = r\,. \tag{15}$$

A little analysis shows that when freefall is initiated arbitrarily close to the horizon ($\varepsilon \to 0$) the initial Schwarzschild time takes the form

$$t_S\left(\text{initial}\right) = t_{PG} + r_c\left(\ln\left(4/\varepsilon\right) - 2\right) - t_R\,.$$

This shows that the proximity to the critical radius of the starting point is limited. That is, it would take an unlimited Schwarzschild time to position an observer at the critical radius in order to commence a freefall to the singularity. Therefore, due to the divergence of the logarithm term the constant t_R would need to be set to an impossibly large value. Hence in practice there is a limit to how close we can position an observer before total evaporation.

However, the discussion so far does not resolve the issue that Kassner's model, for observers starting at sufficiently large radius and/or inward radial velocity, also predicts that the black hole evaporates totally while $r > r_c$. Of Kassner's five freefalling observers let us call one of them observer B with reference to figure 3 above. Later he introduces a sixth that fails to reach the horizon before total evaporation. Again with reference to figure 3, let this be observer A that crosses the Cauchy horizon at finite t_S after observer B crosses the event horizon at $t_S = \infty$. This is the same inconsistency visually represented in figure 3. The inconsistency is in relation to the global time variable, t_S, in $\mathcal{M}$. It should not be possible to have events in $\mathcal{M}$ with arbitrarily large t_S in the causal past of events with smaller t_S. At the very least an evaporation model should be based on a time variable that can be made global throughout $\mathcal{M}$, this appears not to be the case with an evaporation model based on t_{PG}. One question is: is it possible to designate t_{PG} as a global time? Equation (3) certainly does not fit the form of equation (2), so its status as a global time variable on which all clocks throughout the sub-manifold, $\mathcal{M}$, can be related, is questionable.

4. Conclusions

Based purely on treatments in two well-known sets of coordinates a major inconsistency in the traditional model of a fully formed, evaporating black hole has been highlighted. This is in the form of a contradiction between the relative positions of non-spacelike separated events and their designated global time as defined throughout the external open manifold, $\mathcal{M}$. This contradiction can be arrived at in two ways: visually with the use of conformal diagrams, and analytically based on coordinates that are regular at the critical radius along with a model that predicts evaporation in a finite time within those coordinates. Exposure of this inconsistency was based on well-known concepts in general relativity and not on any of the more sophisticated ideas that are circulating at the frontiers of current research.

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